SOLUTION SET

ORIE 630 - HW5

Special thanks to Tim and Dan.

1. The direction d of the unbounded feasible ray is a solution to the system $Ax = 0$, $x \geq 0$, $c^T x < 0$, which is system (II$'$) of the alternative form of the Farkas Lemma. Thus, system (I$'$) must be infeasible, i.e., $A^T y \leq c$ is infeasible, i.e., the dual LP is infeasible. We must verify that the unbounded feasible ray solves (II$'$). This is essentially done in the lecture notes from 9/27.

   Let $d = \begin{pmatrix} -\bar{a}_q \\ e_q \end{pmatrix}$. By the unboundedness criterion we have $\bar{a}_q \leq 0$, so $d \geq 0$. Also, we know that $\bar{c}_q < 0$ from the unboundedness criterion. Then

$$\begin{aligned} Ad &= (B\ N)\begin{pmatrix} -\bar{a}_q \\ e_q \end{pmatrix} \\ &= -B\bar{a}_q + Ne_q \\ &= -BB^{-1}a_q + a_q \\ &= 0. \end{aligned}$$

   Also,

$$\begin{aligned} c^T d &= (c_B^T\ c_N^T)\begin{pmatrix} -\bar{a}_q \\ e_q \end{pmatrix} \\ &= -c_B^T B^{-1} a_q + c_q \\ &= c_q - a_q^T B^{-T} c_B \\ &= c_q - a_q^T \bar{y} \\ &= \bar{c}_q \\ &< 0. \end{aligned}$$

2. Given a current basic feasible solution $\bar{x}$ and reduced costs $\bar{c}_N$, recall the equation that relates the current objective function value $\bar{\zeta}$ to the objective function value ζ for any feasible solution

$$\zeta = \bar{\zeta} + \bar{c}_N^T x_N$$

   If $\bar{c} \geq 0$ then $\bar{x}$ is optimal. Otherwise some $\bar{c}_i < 0$. Let $\alpha = -\bar{c}_k = -\min_i \bar{c}_i > 0$. Since the x_i's sum to 1, we can do no better than increasing x_k by 1, and hence decreasing ζ by α. For

$$\begin{aligned} \zeta &= \bar{\zeta} + \bar{c}_N^T x_N \\ &\geq \bar{\zeta} - \alpha e^T x_N \\ &\geq \bar{\zeta} - \alpha \qquad \text{since } e^T x_N \leq e^T x = 1. \end{aligned}$$

3. Suppose you are solving a standard form LP problem with n variables from a given basic feasible solution, and you know that every basic solution has at most one basic variable zero. Show that the simplex method will either terminate or improve the objective function value within n iterations from any basic feasible solution, and deduce that it will terminate in a finite number of iterations.

Suppose we are at a basic feasible solution, $\bar{x}$. If $\bar{c}_N \geq 0$, then we are at an optimal solution and can terminate. Otherwise the simplex method will choose some q for which $\bar{c}_q < 0$, and then find some p for which $\bar{a}_{pq} > 0$. If no such p exists then the problem is unbounded by the unboundedness criterion, and the simplex method will terminate. If such a p exists where $x_p > 0$, then we will move from one basic feasible solution to another, along a direction where the objective function will improve. Otherwise we have $x_p = 0$, and we remain at the same basic feasible solution, but with p removed from the basis and q added. We will now show that we will never be able to add p back into our basis and remain at the same basic feasible solution.

We know before the change of basis we have

$$0 = \bar{c}_p = c_p - a_p^T \bar{y}.$$

In class we showed that after the change of basis

$$\bar{y}_+ = \bar{y} + \frac{\bar{c}_q}{\bar{a}_{pq}} B^{-T} e_p$$

so the updated reduction cost for p is simply

$$\begin{aligned}
\bar{c}_{p+} &= c_p - a_p^T \bar{y}_+ \\
&= c_p - a_p^T (\bar{y} + \frac{\bar{c}_q}{\bar{a}_{pq}} B^{-T} e_p) \\
&= (c_p - a_p^T \bar{y}) - \frac{\bar{c}_q}{\bar{a}_{pq}} (a_p^T B^{-T}) e_p \\
&= \bar{c}_p - \frac{\bar{c}_q}{\bar{a}_{pq}} e_p^T e_p \\
&= -\frac{\bar{c}_q}{\bar{a}_{pq}}.
\end{aligned}$$

Since we assumed that $\bar{c}_q < 0$ and that $\bar{a}_{pq} > 0$ then we know that the new reduced cost for p must be positive. Therefore p will not be chosen to be placed back into the basis on the next iteration of the simplex method. Furthermore, the above analysis holds whenever

$$\bar{c}_p = c_p - a_p^T \bar{y} \geq 0,$$

so once a variable has achieved a positive reduced cost it will never be chosen under any new basis as long as we remain at the same basic feasible solution. Since every basic feasible solution is assumed to have at most one degenerate variable, then at any basic feasible solution there can be at most $n - m + 1$ variables with value 0, so after $n - m + 1$ iterations, if we remain at the same basic feasible solution then there will no longer be any nonbasic variables with negative reduced costs. Thus we will be at an optimal solution. Alternatively, if we ever move away from the current basic feasible solution then we must have reduced our objective function value. It is also possible that we meet the unboundedness criterion in one of these iterations. Regardless of the case, after $n + m - 1$ iterations we either improve the objective function value or else terminate the simplex method. ∎