

# SOLUTION SET

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10pt  
✓

1. Here's another approach to proving strong duality. By the Fundamental Theorem of LP, (D) is either infeasible, or unbounded, or has an optimal solution  $y_*$ . The proof technique we used takes care of the first two cases: so let us suppose (D) has an optimal solution, with value  $\zeta_*$ . This means that there is no feasible solution to  $A^T y \leq c, b^T y > \zeta_*$ . This is like system (II) in the Farkas Lemma, except that the right-hand sides are not all zero. Modify this system to get a new system like (II) that does not have a feasible solution. Now apply the Farkas Lemma to show that (P) has an optimal solution with value  $\zeta_*$ .

There is no feasible solution to the system, (S):  $A^T y \leq c, b^T y > \zeta_*$ , so the following system, (S'), also has no feasible solution.

$$\begin{aligned} A^T y - c\eta &\leq 0 \\ b^T y - \zeta_*\eta &> 0 \\ \eta &\geq 0 \end{aligned}$$

To see this suppose there is a solution,  $(\hat{y}, \hat{\eta})$  to (S'). If  $\hat{\eta} = 0$  then  $A^T \hat{y} \leq 0$  and  $b^T \hat{y} > 0$ , so

$$A^T(\hat{y} + y_*) = A^T \hat{y} + A^T y_* \leq 0 + c = c$$

and

$$b^T(\hat{y} + y_*) = b^T \hat{y} + b^T y_* > 0 + \zeta_* = \zeta_*$$

thus  $\hat{y} + y_*$  is a solution to (S) which is a contradiction. Alternatively, if  $\hat{\eta} \neq 0$  then

$$\begin{aligned} A^T(\hat{y}/\hat{\eta}) &\leq c \\ b^T(\hat{y}/\hat{\eta}) &> \zeta_* \end{aligned}$$

so  $\hat{y}/\hat{\eta}$  is a solution to (S) which is again a contradiction.

Therefore

$$\begin{bmatrix} A^T & -c \\ 0 & -1 \end{bmatrix} \begin{bmatrix} y \\ \eta \end{bmatrix} \leq \vec{0}, \quad [b^T - \zeta_*] \begin{bmatrix} y \\ \eta \end{bmatrix} > 0$$

has no solution, so by the Farkas Lemma we know

$$\begin{bmatrix} A & 0 \\ -c^T & -1 \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \leq \begin{bmatrix} b \\ -\zeta_* \end{bmatrix}, \quad \begin{bmatrix} x \\ w \end{bmatrix} \geq \vec{0}$$

has a feasible solution. This implies that there is some feasible  $(x, w)$  such that

$$\begin{aligned} Ax &= b \\ c^T x + w &= \zeta_* \\ c, w &\geq 0 \end{aligned}$$

which means there is a feasible solution  $x$  to (P) with  $c^T x \leq \zeta_*$ . However, we know by weak duality that the objective function value of (P) cannot be less than the objective function value for (D), so there is an optimal solution to (P) with value  $\zeta_*$ . ■

**2. (Strict Complementary Slackness)** Suppose that (P) and (D) have optimal solutions with objective values equal to  $\zeta_*$ .

(a) Show that the sets of optimal solutions of both (P) and (D) are convex sets.

(b) By considering the LP problem

$$\min\{-e_j^T x : Ax = b, -c^T x \geq -\zeta_*, x \geq 0\},$$

show that either there is an optimal solution to (P) with its  $j$ th component positive, or there is an optimal solution to (D) with its  $j$ th inequality holding strictly.

(c) Show that there are optimal solutions  $x_*$  and  $y_*$  to (P) and (D) so that, with  $s_* := c - A^T y_*$ ,  $s_* + x_* > 0$ . (These are so-called strictly complementary solutions.)

3pt. (a) Suppose  $x_1$  and  $x_2$  are optimal solutions to (P), with  $c^T x_1 = c^T x_2 = \zeta_*$  and  $Ax_1 = Ax_2 = b$ ,  $x_1, x_2 \geq 0$ . Then for any  $\lambda$  with  $0 \leq \lambda \leq 1$  we have

$$\begin{aligned} c^T(\lambda x_1 + (1 - \lambda)x_2) &= \lambda c^T x_1 + (1 - \lambda)c^T x_2 = \lambda \zeta_* + (1 - \lambda)\zeta_* = \zeta_* \\ A(\lambda x_1 + (1 - \lambda)x_2) &= \lambda Ax_1 + (1 - \lambda)Ax_2 = \lambda b + (1 - \lambda)b = b \\ \lambda x_1 + (1 - \lambda)x_2 &\geq 0, \end{aligned}$$

since  $x_1, x_2, \lambda, (1 - \lambda) \geq 0$ . Therefore  $\lambda x_1 + (1 - \lambda)x_2$  is an optimal solution to (P), so the optimal solutions to (P) form a convex set.

Suppose  $y_1$  and  $y_2$  are optimal solutions to (D), with  $b^T y_1 = b^T y_2 = \zeta_*$  and  $A^T y_1 \leq c$ ,  $A^T y_2 \leq c$ . Then for any  $\lambda$  with  $0 \leq \lambda \leq 1$  we have

$$\begin{aligned} b^T(\lambda y_1 + (1 - \lambda)y_2) &= \lambda b^T y_1 + (1 - \lambda)b^T y_2 = \lambda \zeta_* + (1 - \lambda)\zeta_* = \zeta_* \\ A^T(\lambda y_1 + (1 - \lambda)y_2) &= \lambda A^T y_1 + (1 - \lambda)A^T y_2 \leq \lambda c + (1 - \lambda)c = c, \end{aligned}$$

so  $\lambda y_1 + (1 - \lambda)y_2$  is an optimal solution to (D), thus the optimal solutions to (D) form a convex set.

✓ 4pt (b) For a given  $j$ , if there is an optimal solution to (P) with its  $j$ th component positive then we are done. Otherwise assume there is no such solution, thus if  $x$  is optimal in (P) then  $x_j = 0$ . By considering

$$\min\{-e_j^T x : Ax = b, -c^T x \geq -\zeta_*, x \geq 0\},$$

we observe that every feasible solution to this LP is an optimal solution to (P). Furthermore, since every feasible solution to this problem must have  $x_j = 0$ , and there

is at least one feasible solution, the problem is optimized with objective function value 0. Thus by Strong Duality we know that its dual

$$\max\{b^T y - \zeta_* z : A^T y - cz \leq -e_j, z \geq 0\},$$

is maximized with objective function value 0. We know there must exist an optimal solution  $(y_*, z_*)$  to this problem. If  $z_* = 0$ , then we know  $b^T y_* = 0$  and  $A^T y_* \leq -e_j$ . So for any optimal solution  $\bar{y}$  to (D) and any  $\alpha > 0$  we have

$$A^T(\bar{y} + \alpha y_*) = A^T \bar{y} + \alpha A^T y_* \leq c - e_j,$$

which implies  $\bar{y} + \alpha y_*$  is a feasible solution to (D) and  $A_j^T y_* \leq c_j - 1 < c_j$  so the  $j$ th inequality holds strictly. Also  $\bar{y} + \alpha y_*$  is an optimal solution since

$$b^T(\bar{y} + \alpha y_*) = b^T \bar{y} + \alpha b^T y_* = \zeta_* + 0 = \zeta_*.$$

Now if  $z_* \neq 0$  then let  $\hat{y} = y_*/z_*$ . We have

$$\begin{aligned} b^T y_* - \zeta_* z_* &= 0 \Rightarrow b^T \hat{y} = \zeta_* \\ A^T y_* - cz_* &\leq -e_j \Rightarrow A^T \hat{y} \leq c - e_j/z_* \end{aligned}$$

so  $\hat{y}$  is an optimal solution to (D), and in particular  $A_j^T \hat{y} \leq c_j - \frac{1}{z_*} < c_j$ , so the  $j$ th inequality holds strictly.

- 3pt (c) For each  $j = 1, \dots, n$ , let  $x^{(j)}$  and  $y^{(j)}$  be a pair of optimal solutions to (P) and (D) respectively such that either  $x_j^{(j)} > 0$  or the  $j$ th inequality in (D) holds strictly for  $y^{(j)}$ . We know that these solutions must exist for each  $j$  by part (b). Finally define  $x_*$  and  $y_*$  as

$$\begin{aligned} x_* &= \sum_{j=1}^n \frac{x^{(j)}}{n}, \\ y_* &= \sum_{j=1}^n \frac{y^{(j)}}{n}. \end{aligned}$$

We know that both  $x_*$  and  $y_*$  are optimal solutions to (P) and (D) by part (a) since each is a convex combination of optimal solutions. Thus for each  $j$  either  $c_j - A_j^T y_* > 0$  or  $x_{*j} > 0$ , where  $A_j$  is the  $j$ th column of  $A$ . Therefore if  $s_* := c - A^T y_*$ , then  $s_* + x_* > 0$ . ■

10pt 3. We used the Farkas Lemma to prove strong duality. Suppose we had derived the Strong Duality Theorem another way. Show how you could prove the Farkas Lemma from it.

**Theorem (The Farkas Lemma).** *Exactly one of (I) and (II) below has a feasible solution.*

I.  $Ax = b, \quad x \geq 0,$

II.  $A^T y \leq 0, \quad b^T y > 0.$

*Proof.* Once again we show that we cannot have solutions to both (I) and (II). If  $x$  is a solution to (I) and  $y$  is a solution to (II) then

$$0 < b^T y = y^T b = y^T (Ax) = (y^T A)x = (A^T y)^T x \leq 0,$$

which is a contradiction.

Now suppose there is no solution to (I). Let us consider the primal-dual pair of problems

$$\begin{array}{ll} \min & 0^T x \\ \text{(P)} & Ax = b \\ & x \geq 0. \end{array} \qquad \begin{array}{ll} \max & b^T y \\ \text{(D)} & Ay \leq 0 \end{array}$$

We know that (P) is infeasible since (I) has no solution. By Strong Duality, this implies that (D) is either also infeasible or else is unbounded. However, (D) cannot be infeasible since  $y = 0$  is a feasible solution, so it must be unbounded. Thus for any constant  $\alpha$ , there is a feasible  $y$  for which  $b^T y > \alpha$ . In particular there must be a feasible  $y$  for which  $b^T y > 0$ , which is a feasible solution to (II). ■

✓ 4. I claimed in class that strong duality fails for more general conic programming problems, where the nonnegative orthant is replaced by another closed convex cone. Here is an example, showing that optimal solutions may not exist even if the problem is feasible with bounded objective function value.

Recall that  $C \bullet X$  denotes  $\text{trace}(C^T X)$  for any equally-dimensioned matrices  $C$  and  $X$ . Also note that a  $2 \times 2$  symmetric matrix is positive semidefinite iff its diagonal entries are nonnegative and its determinant is nonnegative.

Consider the problem

$$\begin{array}{ll} \min & \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \bullet X \\ & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \bullet X = 2, \\ & X \text{ is positive semidefinite.} \end{array}$$

(a) Show that this problem is feasible.

(b) Show that any feasible solution has nonnegative objective function value.

(c) Show that there are feasible solutions with arbitrarily small positive objective function value, but none with value 0.

- (a) One feasible solution is

$$X = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

where  $X$  is positive semidefinite, the equality constraint is met, and the objective function value is 1.

- (b) A feasible solution,  $X$ , must be positive semidefinite, and so its diagonal entries must be nonnegative. The objective function value is the value of the first diagonal entry, so it too must be nonnegative.

- (c) For any  $\epsilon > 0$ , there is a solution of the form

$$X = \begin{bmatrix} \epsilon & 1 \\ 1 & \frac{1}{\epsilon} \end{bmatrix},$$

with objective function value  $\epsilon$ . However, every feasible solution must be of the form

$$X = \begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix},$$

for some  $a, b$ , since  $X$  must be symmetric and its non-diagonal entries must sum to 2. Thus the objective function value can never reach 0 since this would imply  $a = 0$ , which would mean the determinant of  $X$  would be  $0 \cdot b - 1 \cdot 1 = -1$  which would mean  $X$  is not positive semidefinite. ■