# ORIE 630 - HW3 -- Solution set

Dan Sheldon

September 23, 2005

(14 points)

1. (a) $B_\infty^\circ = B_1$. In the lecture notes from August 25th, we noted that $||z||_1 = \max\{z^T x : ||x||_\infty \le 1\}$. So

$$\begin{aligned} z \in B_1 &\iff ||z||_1 \le 1 \\ &\iff \max\{z^T x : ||x||_\infty \le 1\} \le 1 \\ &\iff z^T x \le 1, \forall x \text{ s.t. } ||x||_\infty \le 1 \\ &\iff z \in B_\infty^\circ. \end{aligned}$$

✓

$B_1^\circ = (B_\infty^\circ)^\circ = B_\infty$ since $B_\infty$ is convex and contains the origin.

$B_2^\circ = B_2$. This is a special case of part (b).

(b) If $S = \{Mx : x \in \mathbf{R}^n, ||x||_2 \le 1\}$, then $S^\circ = \{z : ||M^T z||_2 \le 1\}$.

- ($\subseteq$) If $z \in S^\circ$, then $z^T y \le 1$, $\forall y \in S$. But $y = Mx$, so $z^T Mx \le 1$, or $(M^T z)^T x \le 1$. This is true for all $x$ such that $||x|| \le 1$, so choose in particular $x = M^T z / ||M^T z||$. This gives us

$$(M^T z)^T \frac{M^T z}{||M^T z||} \le 1 \iff ||M^T z|| \le 1$$

- ($\supseteq$) If $||M^T z|| \le 1$, and $y \in S$, then we have $y = Mx$ and $||x|| \le 1$. Then

$$\begin{aligned} 0 &\le ||M^T z - x||^2 \\ &= (M^T z - x)^T (M^T z - x) \\ &= (M^T z)^T (M^T z) - 2(M^T z)^T x + x^T x \end{aligned}$$

✓

Simplifying, we get that $z^T Mx = z^T y \le \frac{1}{2}(||x||^2 + ||M^T z||^2) \le 1$, so $z \in S^\circ$.

**Note**: by letting $M = I$, we see the proof for part (a) that $B_2^\circ = B_2$.

(c) $\mathcal{N}(A)^\circ = \mathcal{R}(A^T)$. This is a special case of part (d). ✓

(d) If $S = \{x \in \mathbf{R}^n : Ax = b\}$, then $S^\circ = \{A^T y : y \in \mathbf{R}^m, b^T y \le 1\}$. Consider any $z \in \mathbf{R}^n$. Then we can write the LP $\max_x\{z^T x : Ax = b\}$ and its dual $\min_y\{b^T y : A^T y = z\}$. The primal is feasible (by assumption), so by strong duality, either the primal is unbounded, or both the primal and dual are feasible and have equal optimal values.

If the primal is unbounded, then $z^T x \to +\infty$ for some $x \in S$, so $z \notin S^\circ$. ✓

If both are feasible, then

$$\begin{aligned} z \in S^\circ &\iff \max_x\{z^T x : Ax = b\} = \min_y\{b^T y : A^T y = z\} \le 1 \\ &\iff \exists y \in \mathbf{R}^m,\ A^T y = z,\ b^T y \le 1 \\ &\iff z \in \{A^T y : b^T y \le 1\} \end{aligned}$$

**Note**: If $b = 0$ as in part (c), then $S^\circ = \{A^T y : y \in \mathbf{R}^m\} = \mathcal{R}(A^T)$.

(e) Note: I'll use the fact from (g) which I'll prove later. If $S = \{y \in \mathbf{R}^m : A^T y \le 0\}$, then $S^* = \{Ax : x \in \mathbf{R}^n, x \ge 0\}$. ✓

Take $S^*$, since this is a convex cone, $(S^*)^\circ = \{y \in \mathbb{R}^m : (Ax)^T y \leq 0, \forall x \geq 0\}$ → from (g).

$\Rightarrow (S^*)^\circ = \{y \in \mathbb{R}^m : x^T A^T y \leq 0, \forall x \geq 0\} = \{y \in \mathbb{R}^m : A^T y \leq 0\}$

$\Rightarrow (S^*)^\circ = S \Rightarrow S^\circ = S^*$ from the theorem that polar of polar is the set itself.

(f) If $S = \{y \in \mathbb{R}^m : y \geq 0\}$, then $S^\circ = \{z \in \mathbb{R}^m : z \leq 0\}$.

- ($\subseteq$) If $z \in S^\circ$, then $z^T y \leq 0$, $\forall y \geq 0$, by the result from part (g). In particular, $z^T e_i = z_i \leq 0$ for $i = 1, \ldots, m$. That is, $z \leq 0$.
- ($\supseteq$) If $z \leq 0$, $y \geq 0$, then $z^T y \leq 0$.

(g) Suppose $C \subseteq \mathbb{R}^n$ is a convex cone. Show that $C^\circ = \{z \in \mathbb{R}^n : x^T z \leq 0, \forall x \in C\}$. Note that this differs from the defintion of $C^\circ$ only in requiring that $z^T x \leq 0$ instead of $z^T x \leq 1$. We show that if $z \in C^\circ$ then it must be the case that $z^T x \leq 0$.

Suppose $z \in C^\circ, x \in C$. Since $C$ is a convex cone, $\alpha x \in C$, $\forall \alpha \geq 0$. This means $\alpha z^T x \leq 1$, $\forall \alpha \geq 0$. It must be the case that $z^T x \leq 0$.

10pts

2. (a) Suppose there is some $y$ such that $A^T y < 0, b^T y > 0$. Let $\gamma = \min_i c_i$, and let $\beta = \max_i (A^T y)_i < 0$.

If $\gamma \geq 0$, then $\alpha y$ is feasible for all $\alpha \geq 0$, since $\alpha A^T y < 0 \leq \gamma e \leq c$.

If $\gamma < 0$, then $\alpha y$ is feasible for all $\alpha \geq \gamma/\beta$, since $\alpha A^T y \leq \alpha \beta e \leq \gamma e \leq c$.

In either case, we can let $\alpha \to +\infty$, and see that $\alpha b^T y \to +\infty$, so the problem is feasible and unbounded.

(b) We'll convert to problem (I″) from recitation 4 notes. We can rewrite $A^T y < 0, b^T y > 0$, as $\bar{A}^T y < 0$ for $\bar{A} = (A, -b)$. Then the alternate system is (II″),

$$\bar{A}\bar{x} = 0, \quad \bar{x} \geq 0, \quad \bar{x} \neq 0, \quad 0^T \bar{x} \leq 0.$$

Note that $0^T \bar{x} \leq 0$ is always true. Letting $\bar{x} = (x; \alpha)$, we get

$$(A, -b)(x; \alpha) = 0, \quad (x; \alpha) \geq 0, \quad (x; \alpha) \neq 0,$$

or,

$$Ax - \alpha b = 0, \quad (x; \alpha) \geq 0, \quad (x; \alpha) \neq 0.$$

10pts

3. Consider a sequence of points $x_k$ that converges to $x$ such that $x_k \notin C$ for all $k$. Such a sequence exists since $x \in \partial C$, so there are points arbitrarily close to $x$ that are not in $C$. For each such point $x_k$ we can invoke the Separating Hyperplane Theorem (since $C$ is closed and convex) to get a $0 \neq a_k \in \mathbb{R}^n$, $\beta_k \in \mathbb{R}$ such that

$$a_k^T x_k > \beta_k > a_k^T z, \quad \forall z \in C.$$

Assume that $a_k$ and $\beta_k$ are normalized so $||a_k|| = 1$. By the hint, there is some convergent subsequence $a_l$, say it converges to $a$. Then

$$a_l^T x_l > \beta_l \geqslant a_l^T z, \quad \forall z \in C. \tag{1}$$

In particular, since $x \in C$,

$$a_l^T x_l > \beta_l \geqslant a_l^T x.$$

But $a_l \to a$ and $x_l \to x$, so $a_l^T x_l$ and $a_l^T x$ each converge to $a^T x$, and it must be the case that $\beta_l \to \beta = a^T x$. Then from (1), we have

$$a^T z \leq a^T x, \quad \forall z \in C.$$

4. The general form of such a pair of problems is (for $a, b, c \in \mathrm{R}$)

$$\begin{array}{lrcl} \min & cx & & \\ & ax & \geq & b \\ & x & \geq & 0 \end{array} \qquad \begin{array}{lrcl} \max & by & & \\ & ay & \leq & c \\ & y & \geq & 0 \end{array}$$

If we choose $a = 0, b = 1, c = -1$, we can see that neither is feasible

6 pts

$$\begin{array}{lrcl} \min & -x & & \\ & 0 & \geq & 1 \\ & x & \geq & 0 \end{array} \qquad \begin{array}{lrcl} \max & y & & \\ & 0 & \leq & -1 \\ & y & \geq & 0 \end{array}$$

However, if we perturb $a$ by an arbitrary positive amount, e.g. set $a = \epsilon > 0$, then the primal is unbounded by letting $x \to +\infty$ and the dual is infeasible.

$$\begin{array}{lrcl} \min & -x & & \\ & x & \geq & \frac{1}{\epsilon} \\ & x & \geq & 0 \end{array} \qquad \begin{array}{lrcl} \max & y & & \\ & y & \leq & -\frac{1}{\epsilon} \\ & y & \geq & 0 \end{array}$$

Or if we perturb $a$ by an arbitrary negative amount, e.g. set $a = -\epsilon < 0$, then the primal is infeasible and the dual is unbounded by letting $y \to +\infty$.

$$\begin{array}{lrcl} \min & -x & & \\ & x & \leq & -\frac{1}{\epsilon} \\ & x & \geq & 0 \end{array} \qquad \begin{array}{lrcl} \max & y & & \\ & y & \geq & \frac{1}{\epsilon} \\ & y & \geq & 0 \end{array}$$