SOLUTION SET

ORIE 630 - HW10

Thanks to Dan Sheldon

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1. (a) The setup is similar to the original cutting-stock problem. Let $a_j \in \mathbb{R}^m$ represent the jth pattern, where a_{ij} is the number of rolls of width w_i in the jth pattern. Then the total profit for the rolls produced by the jth pattern is $a_j^T p$.
Let x_j be the (integer) number of rolls cut in the jth pattern. Then the formulation of the problem to maximize profits while staying within the sales limits and respecting the limit on large rolls is

$$\begin{array}{rrcl} \max & (A^T p)^T x & & \\ & Ax & \leq & 1.1b, \\ & -Ax & \leq & -.9b, \\ & e^T x & \leq & M, \\ & x & \geq & 0. \end{array}$$

(b) We can solve the relaxation of this problem using column generation if we can easily check for dual feasibility, or find some violated dual constraint. The dual of the problem above is

$$\begin{array}{rrcrcrcl} \min & 1.1b^T y_1 & - & .9b^T y_2 & + & \alpha M & & \\ & A^T y_1 & - & A^T y_2 & + & e\alpha & \geq & A^T p \\ & y_1, & & y_2, & & \alpha & \geq & 0. \end{array}$$

A trial dual solution $(\bar{y}_1; \bar{y}_2; \bar{\alpha})$ satisfies the jth constraint as long as

$$a_j^T \bar{y}_1 - a_j^T \bar{y}_2 + \bar{\alpha} \geq a_J^T p,$$

i.e, whenever,

$$(p - \bar{y}_1 + \bar{y}_2)^T a_j \leq \alpha.$$

We can check this for all j by maximizing $(p - \bar{y}_1 + \bar{y}_2)^T a$ over all valid patterns a, and comparing the result with α, that is, by solving the integer knapsack problem $\max\{(p - \bar{y}_1 + \bar{y}_2)^T a : w^T a \leq W,\ a \geq 0\}$. If this uncovers some pattern a for which the optimum value exceeds α, we can add a to the basis.

2. (a) The dual of (P) is

$$\begin{array}{rrcrcl} \max & b_1^T y_1 & + & b_2^T y_2 & & \\ & A_1^T y_1 & & & \leq & c_1, \\ & & & A_2^T y_2 & \leq & c_2, \\ & F_1^T y_1 & + & F_2^T y_2 & \leq & d, \\ & y_1 & & y_2 & \geq & 0. \end{array}$$

This is block-angular, so we can write our master problem in the usual way, with $Q_j = \{y_j : A_j^T y_j \leq c_j\}$ for $j = 1, 2$.

$$\begin{array}{rlclcll} \min & \sum_j [\sum_h (b_j^T v_{jh}) \lambda_{jh} & + & \sum_i (b_j^T d_{ji}) \mu_{ji}] & & & \\ & \sum_j [\sum_h (F_j^T v_{jh}) \lambda_{jh} & + & \sum_i (F_j^T d_{ji}) \mu_{ji}] & \leq & d & \\ & & & e^T \lambda_j & = & 1 & \forall j \\ & & & \lambda_j, \mu_j & \geq & 0 & \forall j \end{array}$$

If $\bar{w} = (\bar{w}_0; \bar{z})$ is the trial dual solution to the master problem, then the dual constraint corresponding to variable λ_{jh} is violated if $(F_j^T v_{jh})^T \bar{w}_0 - \bar{z}_j \leq b_j^T v_{jh}$, i.e., if $(b_j - F_j \bar{w}_0)^T v_{jh} \geq -\bar{z}_j$. So the jth subproblem is

$$\begin{array}{rrcl} \max & (b_j - F_j \bar{w}_0)^T y_j & & \\ & A_j^T y_j & \leq & c_j, \\ & y_j & \geq & 0. \end{array}$$

The dual of the jth subproblem is

$$\begin{array}{rrcl} \min & c_j^T x_j & & \\ & A_j^T x_j & \geq & b_j - F_j \bar{w}_0, \\ & x_j & \geq & 0. \end{array}$$

The interpretation of this is that the jth division of the corporation seeks to meet its goals while minimizing cost, but the corporation is contributing $F_j \bar{w}_0$ towards meeting the goals.

(b) We can write this in block-angular form as follows:

$$\begin{array}{rcccccccl} \min & c_1^T x_1 & +\frac{1}{2} d^T w_1 & + & c_2^T x_2 & +\frac{1}{2} d^T w_2 & & & \\ & A_1 x_1 & + F_1 w_1 & & & & & \geq & b_1, \\ & & & & A_2 x_2 & + & F_2 w_2 & \geq & b_2, \\ & & w_1 & & & - & w_2 & = & 0, \\ & x_1, & & & x_2 & & & \geq & 0. \end{array}$$

We write the master problem in standard fashion, and, if $\bar{y} = (\bar{y}_0, \bar{z})$ is the trial dual solution, the two subproblems are

$$\begin{array}{rcccl} \min & c_1^T x_1 & +\frac{1}{2} (d - \bar{y}_0)^T w_1 & & \\ & A_1 x_1 & + \quad F_1 w_1 & \geq & b_1, \\ & x_1 & & \geq & 0, \end{array}$$

and

$$\begin{array}{rcccl} \min & c_2^T x_2 & +\frac{1}{2} (d + \bar{y}_0)^T w_2 & & \\ & A_2 x_2 & + \quad F_2 w_2 & \geq & b_2, \\ & x_2 & & \geq & 0. \end{array}$$

(c) We imagine (P) representing a problem for a corporation with two divisions. Each division has its own decision variables, and the corporation has decision variables. Each division needs to meet certain goals, with help (or perhaps hindrance) from the corporation.

In each decomposition, the subproblems demonstrate that the divisions can, to some extent, act independently to achieve the optimum, as long as they receive the appropriate direction from the corporation.

In the decomposition of part (a), the corporation directs the divisions by modifying the goals for the division, or, alternately, providing resources towards meeting the goals. Hence it is a "resource-directed" decomposition.

In the decomposition of part (b), the corporation directs the divisions by modifying prices on the division's items based on the extent to which they help meet corporate constraints. Hence, the decomposition is "price-directed". This decomposition has larger subproblems than the resource-directed decomposition.

3. (a) Let $A = \text{diag}(a)$ be the diagonal matrix with the entries of a along the diagonal, then $a^T = e^T A$. Suppose $x \geq 0$ and $a^T x \leq 1$. Then $Ax \geq 0$, since A is diagonal with all positive entries, and $e^T(Ax) = a^T x \leq 1$. Hence $Ax \in S^m$. Furthermore, A is nonsingular because it is diagonal with nonzero diagonal entries. Therefore, A is a nonsingular linear transformation taking $\{x \in R^m : a^T x \leq 1,\, x \geq 0\}$ into S^m.

(b) The piece containing the origin is $\bar{S}^m := \{x \in \mathrm{R}^m : e^T x \leq m/(m+1)\}$. Then $x \in \bar{S}^m \iff x = m/(m+1)Iw$ for some $w \in S^m$. So

$$
\begin{aligned}
\text{vol } \bar{S}^m &= \int_{x \in \bar{S}^m} 1 dx \\
&= \int_{w \in S^m} 1 \cdot \left| \det \frac{m}{m+1} I \right| dw \\
&= \left(\frac{m}{m+1} \right)^m \int_{w \in S^m} dw \\
&= \left(\frac{m}{m+1} \right)^m \text{vol } S^m
\end{aligned}
$$

Assume $m \geq 1$. To see that $(m/(m+1))^m$ is at least $1/e$, note that

$$
\ln(m/(m+1))^m = m(\ln m - \ln(m+1)). \tag{1}
$$

Then, using the Taylor expansion,

$$
\begin{aligned}
\ln(m+1) &= \ln m + \frac{1}{m} - \frac{1}{2m^2} + \frac{1}{3m^3} - \frac{1}{4m^4} + \frac{1}{5m^5} - \cdots \\
&= \ln m + \frac{1}{m} - \underbrace{\frac{1}{m^2} \left(\frac{1}{2} - \frac{1}{3m} \right)}_{\geq 0} - \underbrace{\frac{1}{m^4} \left(\frac{1}{4} - \frac{1}{5m} \right)}_{\geq 0} - \cdots \\
&= \ln m + \frac{1}{m} - \epsilon,
\end{aligned}
$$

where $\epsilon \geq 0$. Subsitituting back into (1),

$$
\begin{aligned}
\ln(m/(m+1))^m &= m(\ln m - \ln m - \frac{1}{m} + \epsilon) \\
&= -1 + \epsilon m \\
&\geq -1,
\end{aligned}
$$

so $(m/(m+1))^m \geq e^{-1}$.