OR630 - Homework Solutions 1

Note: Special thanks to Tim, Dan, and Mingbo who typed the answers and allowed me to share them.

1. (a) Let e be the vector of all ones. Write $w = u - \alpha e$, $u \in \mathbb{R}^n, u \geq 0, \alpha \geq 0$. The u_i correspond to the w_i, but are nonnegative, and there is one additional (nonnegative) variable α. To see that any $w \in \mathbb{R}^n$ can be written this way, consider an arbitrary $w \in \mathbb{R}^n$. If $w \geq 0$, then set $\alpha = 0$ and $u = w$. Otherwise, set $\alpha = -\min_i w_i$, and $u = w + \alpha e$. For any i we have $w_i \geq -\alpha$ by our choice of α, so $u_i = w_i + \alpha \geq 0$. 3.5P

   (b) Again, let e be the vector of all ones. Write the set of equality constraints $A_w^T y = c_w$ as the following inequality constraints:

$$\begin{aligned} A_w^T y &\leq c_w \\ \sum_i (c_w)_i &\leq \sum_i (A_w^T y)_i \end{aligned}$$

3.5P

It's clear that these are equivalent to the equality constraints. There is only one additional constraint (the final one) and it can be rewritten as $e^T c_w \leq e^T A_w^T y$. Then we can rewrite the LP problem as

$$\begin{array}{lrcl} \max & b^T y & & \\ & A_w^T y & \leq & c_w, \\ & -e^T A_w^T y & \leq & -e^T c_w, \\ & A_x^T y & \leq & c_x \end{array}$$

Given this formulation, it's easy to see that this technique is the "dual" of the technique in (a). We take the dual of this LP, using $u \in \mathbb{R}^n, \alpha \in \mathbb{R}$, and $x \in \mathbb{R}^n$, respectively, to weight the constraints of the orignal LP. This gives us the dual LP

$$\begin{array}{lcccccccc} \min & c_w^T u & - & \alpha c_w^T e & + & c_x^T x & & \\ & A_w u & - & \alpha A_w e & + & A_x x & = & b \\ & & & & & u, \alpha, x \geq 0 & & \end{array}$$

which is the LP from part (a), but with w replaced by $u - \alpha e$, and all nonnegative variables.

c) Consider a more general primal problem (P)

3P

$$\begin{aligned} \min_{x,w,u} \ & c_x^T x + c_w^T w + c_u^T u \\ & A_x x + A_w w + A_u u = b \\ & \bar{A}_x x + \bar{A}_w w + \bar{A}_u u \geq \bar{b} \\ & A'_x x + A'_w w + A'_u u \leq b' \\ & x \geq 0, u \leq 0, w \text{ unrestricted.} \end{aligned}$$

We can take arbitrary multiples of the equality constraints, nonnegative multiples of the greater-than-or-equal-to constraints and the non-positive multiples of the less-than-or-equal-to constraints to get

$$(A_x^T y + \bar{A}_x^T z + A_x'^T v)x + (A_w^T y + \bar{A}_w^T z + A_w'^T v)w + (A_u^T y + \bar{A}_u^T z + A_u'^T v)u \geq b^T y + \bar{b}^T z + b'^T v$$

as long as $z \geq 0$, $v \leq 0$. Since $x \geq 0$, $u \leq 0$ and w unrestricted, then we need $A_x^T y + \bar{A}_x^T z + A_x'^T v \leq c_x$, $A_w^T y + \bar{A}_w^T z + A_w'^T v = c_w$ and $A_u^T y + \bar{A}_u^T z + A_u'^T v \geq c_u$, so that whenever (y, z, v) satisfies

$$\begin{aligned}
&A_x^T y + \bar{A}_x^T z + A_x'^T v \leq c_x \\
&A_w^T y + \bar{A}_w^T z + A_w'^T v = c_w \\
&A_u^T y + \bar{A}_u^T z + A_u'^T v \geq c_u \\
&z \geq 0, v \leq 0
\end{aligned}$$

we have

$$\begin{aligned}
&c_x^T x + c_w^T w + c_u^T u \\
&\geq (A_x^T y + \bar{A}_x^T z + A_x'^T v)x + (A_w^T y + \bar{A}_w^T z + A_w'^T v)w + (A_u^T y + \bar{A}_u^T z + A_u'^T v)u \\
&\geq b^T y + \bar{b}^T z + b'^T v
\end{aligned}$$

for any feasible solution to (P). Then the best possible bound is the solution to the dual problem (D),

$$\begin{aligned}
\max_{y,z,v} \quad & b^T y + \bar{b}^T z + b'^T v \\
& A_x^T y + \bar{A}_x^T z + A_x'^T v \leq c_x \\
& A_w^T y + \bar{A}_w^T z + A_w'^T v = c_w \\
& A_u^T y + \bar{A}_u^T z + A_u'^T v \geq c_u \\
& z \geq 0, v \leq 0, y \text{ unrestricted}
\end{aligned}$$

In fact, the procedure shown above is the weak duality. Then by examining the weak duality as shown, we can conclude that for every non-positive primal variable there is a greater-than-or-equal-to dual constraint; for every less-than-or-equal-to primal constraint there is an non-positive dual variable.

(10 points)

II. PROBLEM 2

Let $e \in \mathbb{R}^n$ be a vector of 1. Then we have $\|Aw - b\|_\infty \leq \eta$ if and only if $e\eta - (Aw - b) \geq 0$ and $e\eta - (b - Aw) \geq 0$. Therefore, the problem of minimizing $\|Aw - b\|_\infty$ is equivalent to minimizing η constrained by $e\eta - (Aw - b) \geq 0$ and $e\eta - (b - Aw) \geq 0$. Therefore, we can formulate the LP problem in primal form (P) as the following:

$$\begin{aligned} \min_{w \in \mathbb{R}^n, \eta \in \mathbb{R}} \quad & \eta \\ & e\eta + Aw \geq b, \\ & e\eta - Aw \geq -b, \\ & \eta \geq 0. \end{aligned}$$

Then the dual of (P) is

$$\begin{aligned} \max \quad & b^T y - b^T z \\ & e^T y + e^T z \leq 1 \\ & A_w^T y - A_w^T z = 0 \\ & y, z \geq 0 \end{aligned}$$

After simplification by setting $\lambda = y - z$, we can get

$$\begin{aligned} \max_{\lambda \in \mathbb{R}^n} \quad & b^T \lambda \\ & \|\lambda\|_1 \leq 1 \\ & A^T \lambda = 0 \end{aligned}$$

which involves 1-norm.

Question 3:

The dual of this problem is simply $\max\{b^t y : A^T y = c\}$. To show strong duality, there are three cases we must consider:

i *Neither problem has a feasible solution.* In this case neither problem has an optimal solution either.

ii *One problem has a feasible solution but the other does not.* Without loss of generality let us assume that it is the primal problem that has a feasible solution. Then we know $A^T y \neq c$ for all y. Thus c is not contained in the column space of A^T The feasible region for the primal problem must be an affine subspace of some dimension greater than 1 (since A does not span $\mathbb{R}^n$), which c is not contained in. Hence there is a component of c that is orthogonal to the feasible region of the primal problem. Since this is a affine subspace, this means that we can move x in some direction indefinitely where the objective function is increasing. Therefore the objective function is unbounded and there is no optimal solution.

iii *Both problems have feasible solutions.* In this case if x and y are the feasible solutions then

$$b^T y = y^T b = y^T(Ax) = (y^T A)x = (A^T y)^T x = c^T x,$$

so the two solutions must be equal.

So any feasible solution is optimal.

■

4. Let S^n denote the space of real symmetric $n \times n$ matrices. This is a finite-dimensional vector space: indeed, by considering just hte upper-triangular entries or by taking an appropriate basis, it can be viewed as isomorphic to $\mathbb{R}^{n(n+1)/2}$. A matrix $A \in S^n$ is called *positive semidefinite* if $x^T Ax \geq 0$ for all $x \in \mathbb{R}^n$.
Show that the set of positive semidefinite real symmetric $n \times n$ matrices is a convex cone containing the origin. You can do this directly, or by showing that it is an (infinite) intersection of half-spaces corresponding to hyperplanes through the origin.

To show that the set of positive semidefinite real symmetric $n \times n$ matrices is a convex cone we simply must show that the set is closed under finite positive linear combinations. Let $A_1, \ldots, A_n \in S^n$ be positive semidefinite and $\lambda_1, \ldots, \lambda_n \geq 0$. Then for any $x \in \mathbb{R}^n$ we have

$$\begin{aligned} x^T \left(\sum_{i=1}^n \lambda_i A_i \right) x &= \left(\sum_{i=1}^n x^T \lambda_i A_i \right) x \\ &= \left(\sum_{i=1}^n x^T \lambda_i A_i x \right) \\ &= \left(\sum_{i=1}^n \lambda_i x^T A_i x \right) \\ &= \left(\sum_{i=1}^n \lambda_i c_i \right), \geq 0 . \end{aligned}$$

where the c_i are nonnegative constants, which we know to be true by the definition of positive semidefinite. Since the λ_i are positive, we know this sum must be nonnegative, thus any positive linear combination of positive semidefinite matrices is also positive semidefinite. Furthermore if $A = 0$ then $x^T Ax = 0 \geq 0$ for all $x \in \mathbb{R}^n$, so the set contains the origin. ■

We also know that if $A_i \in S^n$ are symmetric, then $\sum_{i=1}^n \lambda_i A_i$ is also symmetric.

$\Rightarrow \forall A_i \in S^n \Rightarrow \sum_{i=1}^n \lambda_i A_i \in S^n$ for $\lambda_i \geq 0$. ■

Also note that each set $\{A : x^T A x \geq 0\}$ is a halfspace with 0 in its boundry, so a convex cone, and the set of psd matrices is exactly the intersection of all these.