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| **Mathematical Programming II** | **Lecture 24** |
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## The story of the Pessimist vs. Optimist (Robust optimization)

We'll consider linear programming problems where some or all of the data are uncertain. We want a feasible solution with good objective value for any data in a specified set. Consider

$$\begin{aligned} \max \quad & b^T y \\ & A^T y \le c. \end{aligned}$$

This is equivalent to

$$\begin{aligned} \max \quad & \eta \\ & \eta - b^T y \qquad \le 0 \\ & \quad A^T y - c\xi \le 0 \\ & \qquad\qquad \xi \le 1 \\ & \qquad\quad -\xi \le -1, \end{aligned}$$

where all of the data is in the constraint matrix. So we have

$$\begin{aligned} \max \quad & b^T y \\ & A^T y \le c, \end{aligned}$$

with only uncertainty in $A$. We'll assume each column $a_j$ of $A$ lies in an uncertainty set $E_j$. Conservative thinking leads to the problem

$$\begin{aligned} \max \quad & b^T y \\ & a_j^T y \le c, \qquad \forall a_j \in E_j, j = 1, \cdots, n. \end{aligned}$$

This is a so-called semi-infinite LP problem and can model nonlinear problems: an infinite number of linear constraints can model a quadratic constraint, for instance (see the example below).

We'll model the $E_j$'s as ellipsoids, possibly degenerate (e.g., each $a_j$ may be sparse and zero coefficients are probably "certain"). Let $E_j = \{\bar{a}_j + D_j w_j : \|w_j\|_2 \le 1\}$ for some $\bar{a}_j \in \mathbb{R}^m$, $D_j \in \mathbb{R}^{m \times p_j}$. Fix $j$; then we want:

$$\max\{a_j^T y : a_j \in E_j\} \le c_j,$$

a single constraint. But

$$\begin{aligned} & \max\{a_j^T y : a_j \in E_j\} \\ & = \max\{\bar{a}_j^T y + (D_j^T y)^T w_j : \|w_j\|_2 \le 1\} \\ & = \bar{a}_j^T y + \|D_j^T y\|_2. \end{aligned}$$

So we get the deterministic equivalent of our robust LP program:

$$\begin{array}{ll} \max & b^T y \\ & \bar{a}_j^T y + \|D_j^T y\|_2 \le c_j, \qquad j = 1, \cdots, n. \end{array}$$

Write this as the conic programming problem:

$$\begin{array}{llll} \max & b^T y & & \\ & \bar{a}_j^T y + \xi_j & = c_j, & j = 1, \cdots, n \\ & D_j^T y & + z_j = 0, & j = 1, \cdots, n \\ & y \in \mathbf{R}^m, \begin{pmatrix} \xi_j \\ z_j \end{pmatrix} \in K_2^{1+p_j}, & & j = 1, \cdots, n. \end{array}$$

This is a second-order cone problem. If all the $p_j$'s are small, this is almost as easy to solve as a comparable LP problem. The dual of this conic problem is

$$\begin{array}{ll} \min & c^T x \\ & \sum_{j=1}^{n} (\bar{a}_j x_j + D_j v_j) = b \\ & \begin{pmatrix} x_j \\ v_j \end{pmatrix} \in (K_2^{1+p_j})^* = K_2^{1+p_j}, \qquad j = 1, \cdots, n. \end{array}$$

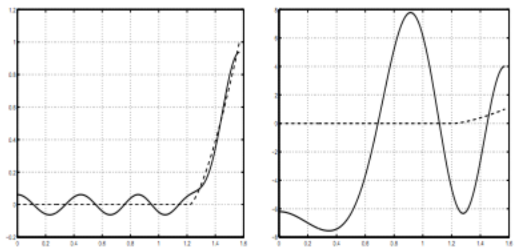
Write $v_j = x_j w_j$, $w_j \in \mathbf{R}^{p_j}$, so $\|w_j\|_2 \le 1$. Then we get

$$\begin{array}{ll} \min\limits_{x, w_1, \cdots, w_n} & c^T x \\ & \sum_{j=1}^{n} (\bar{a}_j + D_j w_j) x_j = b \\ & x \ge 0, \|w_j\| \le 1, \forall j. \end{array}$$

This is equivalent to

$$\begin{array}{ll} \min & c^T x \\ & \sum_{j=1}^{n} a_j x_j = b \\ & x \ge 0, \end{array}$$

for *some* $a_j \in E_j$, $j = 1, 2, \cdots, n$. This is the optimist's problem: $x$ is feasible as long as it is feasible for some data in the uncertainty set. A nice example of designing an antenna via robust LP



**“Dream and reality”:** the nominal (left, solid) and an actual (right, solid) diagrams
[dashed: the target diagram]

is in [1] *Page 101.*

Remarks on semi-infinite programming:

$$(P) \qquad \begin{array}{ll} \max & b^T y \\ & a(t)^T y \le c(t), t \in T, \end{array}$$

where $T$ is compact.

**Example 1** $T = [0, 2\pi]$, $a(t) = (\cos(t); \sin(t))$, $c(t) = 1$*; then the feasible region is the unit ball in* $\mathbf{R}^2$*.*

Dual:

$$(D) \qquad \begin{array}{ll} \min & \text{“}\sum_{t \in T} c(t)x(t)\text{”} = \int c(t)x(t)dt \\ & \text{“}\sum_{t} a(t)x(t)\text{”} = \int a(t)x(t)dt = b \\ & x(t) \ge 0,\ t \in T. \end{array}$$

Think of the dual simplex algorithm for $(P)$, which chooses just $m$ of the $a(t)$'s with $b$ a nonnegative combination. So $x$ would be a discrete measure concentrating on just $m$ points. Then somehow find a violated constraint $t$ for the corresponding $y$. As the iterations proceed, $a(t_1), \cdots, a(t_m)$ may become degenerate, e.g., for $m = 2$, $[a(t_1), a(t_2)]$ becomes ill-conditioned as $|t_1 - t_2| \to 0$. Instead, we could consider a basis of $[a(t_1), \frac{a(t_2) - a(t_1)}{t_2 - t_1}]$ or in the limit $[a(t_1), a'(t_1)]$. See [2] for details.

# References


[1] A. Ben-Tal, A.S. Nemirovski, *Lectures on Modern Convex Optimization*

[2] E.J. Anderson, A.S. Lewis, *An Extension of the Simplex Algorithm for Semi-infinite Linear Programming*, 1988