Mathematical Programming II	**Lecture 10**
ORIE 6310 Spring 2014	**February 27, 2014**
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Consider

$$
\begin{aligned}
w &= dz_0 + Mz + q && (1)\\
Iv &+ (-q)\, y_0 + (-M)\, y = d. && (2)
\end{aligned}
$$

If w_J, z_0 and z_K is a basic solution to (1), then v_J, y_0 and y_K is a basic solution to (2). Here $J \cup K \cup \{l\}$ is a partition of N and l is the missing index.

Remember that $L = K \cup \{l\}, H = \{0\} \cup K, F = [q, M]$ and

$$
\begin{aligned}
B &= \begin{bmatrix} I_{JJ} & -F_{JH} \\ 0 & -F_{LH} \end{bmatrix},\\
B^{-1} &= \begin{bmatrix} I_{JJ} & -F_{JH}F_{LH}^{-1} \\ 0 & -F_{LH}^{-1} \end{bmatrix}.
\end{aligned}
$$

Lemma 1 *We have a feasible basic solution corresponding to the partition $J \cup K \cup \{l\} = N$ iff F_{LH} is nonsingular, and $B^{-1}\begin{bmatrix} \delta^n \\ \vdots \\ \delta \end{bmatrix} \geq 0$ for all sufficiently small positive δ.*

Note if i is the last index in L, then

$$
B^{-1}[e_n, \ldots, e_1] = \left[\begin{array}{ccc|c|cc}
0 & 0 & 0 & & & \\
\vdots & \vdots & \vdots & & & \\
0 & 0 & 0 & & \cdots & \cdots \\
0 & 0 & 1 & & & \\
0 & 1 & 0 & & & \\
1 & 0 & 0 & & &
\end{array}\right],
\qquad \underset{B^{-1}e_i}{\uparrow}
$$

and we need this to be lexicographically positive (the first nonzero entry in each row is positive) for feasibility. Here the first few columns represent columns of the identity for $j \in J$, $j > i$.

Probability distribution We'll assume that (M, q) is generated from some distribution satisfying

a. [Nondegeneracy] With probability 1:

(i) If J and K are subsets of N of the same cardinality with $|J \setminus K| \leq 1$, then M_{JK} is nonsingular;

(ii) Any a.c. solution to $w = Mz + q$ (nonnegative or not) has at least n nonzero components.

b. (Sign invariance) For any "sign" matrix S (diagonal with diagonal entries $s_i := s_{ii}$ with $s_{ii} \in \{-1, 1\}$)

$$(SMS, Sq) \sim (M, q),$$

i.e., they have the same distribution.

Note: $P-$matrices and monotone matrices are invariant under these sign switches.

If $s_i = -1$, we could keep M and q the same and change $w_i, z_i \geq 0$ to $w_i, z_i \leq 0$.

A simple case: choose M, q satisfying (a) (i) and (ii), then choose each (SMS, Sq) w.p. 2^{-n}. If we can obtain a bound for this case, it will hold for all such distributions.

Lemma 2 *If (M, q) satisfies (a) (i) and (ii), then for any partition $J \cup K \cup \{l\} = N$, F_{LH} is nonsingular.*

Proof:
$F_{LH} = [q_L, M_{LK}]$. Suppose there is a nontrivial linear dependence among the columns of F_{LH}.

(1) If the dependence does not involve q, then there is a nontrivial linear dependence among the columns of M_{LK}, so among the columns of $[M_{Ll}, M_{LK}] = M_{LL}$. This contradicts (a) (i).

(2) Suppose the linear dependence involves q_L. Then scale to get $q_L + M_{LK}z_K = 0$. Then set

$$\begin{aligned} w_J &:= q_J + M_{JK}z_K \\ w_L &:= q_L + M_{LK}z_K = 0. \end{aligned}$$

Set $z_J = 0$, and then we get $w = Mz + q$, a.c., but with only $n - 1$ nonzeroes. This contradicts (a) (ii).

$\square$

Lemma 3 *If (M, q) satisfies (a) (i) and (ii), then for any partition $J \cup K \cup \{l\} = N$, all entries of $B^{-1}e_i$ (i the last index in L) are nonzero.*

Proof: Recall

$$B = \begin{bmatrix} I_{JJ} & -F_{JH} \\ 0 & -F_{LH} \end{bmatrix}.$$

Look at $e_m^\top B^{-1} e_i$ for all $m \in J \cup H = J \cup K \cup \{0\}$.

(1) $m = 0$. Let

$$e_i = \begin{bmatrix} I_{JJ} & -F_{JH} \\ 0 & -F_{LH} \end{bmatrix} \begin{pmatrix} v_J \\ y_0 \\ y_K \end{pmatrix}.$$

If $e_0^\top B^{-1} e_i$ is zero, then $y_0 = 0$. So

$$\begin{aligned} 0 &= v_J - M_{JK} y_K \\ 0 &= -M_{L\setminus\{i\}K} y_K \\ 1 &= -M_{iK} y_K \Rightarrow y_K \neq 0; \end{aligned}$$

then $M_{L\setminus\{i\}K}$ is singular and this contradicts (a) (i).

(2) $m = j$: Suppose $e_j^\top B^{-1} e_i = 0$. Then

$$e_i = \begin{bmatrix} I_{JJ} & -F_{JH} \\ 0 & -F_{LH} \end{bmatrix} \begin{pmatrix} v_J \\ y_0 \\ y_K \end{pmatrix}$$

with $v_j = 0$. Scale so that $y_0 = 1$ (by the case above we know y_0 is nonzero) to get w_J, 1, and z_K. Then

$$\begin{aligned} w_J - q_J - M_{JK} z_K &= 0 \\ -q_{L\setminus\{i\}} - M_{L\setminus\{i\}K} z_K &= 0. \end{aligned}$$

Set $w_i := q_i + M_{iK} z_K$, $w_{L\setminus\{i\}} := 0$ and note that $w_j = 0$. Also set $z_J := 0$ and $z_l := 0$. Then we have an a.c. solution with only $n-1$ nonzeroes. This contradicts (a) (ii).

Similarly we can show $e_k^\top B^{-1} e_i \neq 0$ for $k \in K$.

□

Let us look at the effects of sign switches on B and B^{-1}. Suppose M and q change to SMS and Sq. Then F becomes $\widetilde{F} = SF \begin{bmatrix} 1 & 0 \\ 0 & S \end{bmatrix}$ and so

$$\begin{aligned} \widetilde{F}_{JK} &= S_{JJ} F_{JH} S_{HH}, \\ \widetilde{F}_{LH} &= S_{LL} F_{LK} S_{HH} \end{aligned}$$

where S_{HH} is the appropriate submatrix of $\begin{bmatrix} 1 & 0 \\ 0 & S \end{bmatrix}$, or $\begin{bmatrix} 1 & 0 \\ 0 & S_{KK} \end{bmatrix}$. So we have

$$\begin{aligned} \widetilde{F}_{LH}^{-1} &= S_{HH} F_{LH}^{-1} S_{LL}, \\ \widetilde{F}_{JH} \widetilde{F}_{LH}^{-1} &= S_{JJ} F_{JH} F_{LH}^{-1} S_{LL}. \end{aligned}$$

Lemma 4 *Consider the basis corresponding to the partition $J \cup K \cup \{l\}$, and let i denote the last index in $L := K \cup \{l\}$. Then if $l = i$, the basis is feasible with probability 2^{-i}, while if $l \neq i$, the basis is feasible with probability at most 2^{-i+1}.*

Proof:

The jth entry ($j \in J$) of $\widetilde{B}^{-1}e_i$ is $s_j \left(e_j^\top B^{-1} e_i\right) s_i$, positive with probability 1/2, switching signs with s_j.

The 0th entry of $\widetilde{B}^{-1}e_i$ is $\left(e_0^\top B^{-1} e_i\right) s_i$, positive with probability 1/2, switching signs with s_i.

The kth entry ($k \in K, k \neq i$) is $e_k^\top \widetilde{B}^{-1} e_i = s_k \left(e_k^\top B^{-1} e_i\right) s_i$, positive with probability 1/2, switching signs with s_k.

If $l = i$, this accounts for all the components; we need the components in $\{1, \ldots, i\}$ to be positive, and this happens with probability 2^{-i} (all the s's named above are independent).

If $l \neq i$, then we have analyzed all but the ith component, which is $e_i^\top \widetilde{B}^{-1} e_i = s_i \left(e_i^\top B^{-1} e_i\right) s_i$, which doesn't change sign with S. If it's negative, the basis is infeasible; if it is positive, the basis is feasible w.p. 2^{-i+1}, since the probability is 1/2 for all components in $\{1, \ldots, i-1\}$, and these events are all independent. □

Theorem 1 *If (M, q) is generated from a distribution as above, the lexicographic Lemke algorithm will take an expected number of steps at most $\frac{n(n+1)}{4}$.*

Proof:

The number of steps is bounded by the number of a.c. feasible solutions, i.e., the number of partitions J, K, l giving feasibility.

The index i runs from 1 to n. If $l = i$, then K is some subset of $\{1, \ldots, i-1\}$; there are 2^{i-1} such choices and then J is fixed. For any such partition, the probability of feasibility is 2^{-i} by the lemma. If $l < i$, then there are $i - 1$ choices for l, and then 2^{i-2} choices for K (any subset of $\{1, \ldots, i-1\} \setminus \{l\}$, plus $\{i\}$). Then J is determined, and by the lemma the probability of feasibility is at most 2^{-i+1}.

So the expected number of feasible a.c. solutions is at most

$$\sum_{i=1}^{n} \left(2^{i-1}\frac{1}{2^i} + (i-1)2^{i-2}\frac{1}{2^{i-1}}\right) = \sum_{i=1}^{n} \left(\frac{1}{2} + \frac{i-1}{2}\right) = \frac{n(n+1)}{4}.$$

□