Mathematical Programming II	Lecture 9
ORIE 6310 Spring 2014	February 25, 2014
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Polynomial expected behavior of a pivoting algorithm:

- Class of problems;
- Variant algorithm;
- Probability distribution;
- Analysis.

–**The general linear complementarity problem**.
–**The lexicographic Lemke (LL) algorithm**:
This is Lemke's algorithm (complementary pivot algorithm 2) with $d = \begin{pmatrix} \delta^n \\ \cdots \\ \delta \end{pmatrix}$, $0 < \delta \ll 1$.

Consider this for two cases:

- M is a P-matrix; then recall the LCP has a unique complementary solution for all q. Think of $z_0 = \delta^{-n+j-0.5}, 0 \leq j \leq n$. As j goes from 0 to n, we get the complementary solutions to all the LCP subproblems defined by the first j rows and j columns of M (also giving a P-matrix) and the first j rows of q. We are (almost) not perturbing the first j entries of q and making the others $+\infty$, so $w_k = +\infty$, $z_k = 0$, for $k > j$. This doesn't work for LP, since the corresponding LCP has a block of zeros in the top left. So it has no complementary solution to such subproblems in general.

- For LP, consider the linear complementarity problem associated with the LP problems:

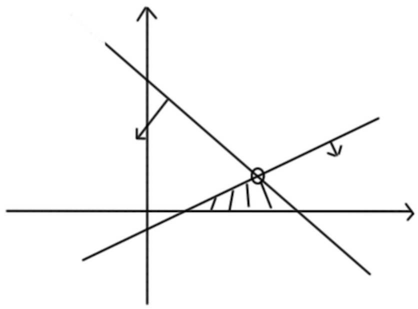
$$(P) \qquad \begin{aligned} \min \quad & c^T x \\ & Ax \geq b \\ & x \geq 0, \end{aligned}$$

$$(D) \qquad \begin{aligned} \max \quad & b^T y \\ & A^T y \leq c \\ & y \geq 0. \end{aligned}$$

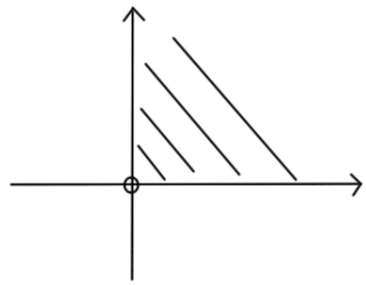
  Then, we have $M = \begin{pmatrix} 0 & -A^T \\ A & 0 \end{pmatrix}$ and $q = \begin{pmatrix} c \\ -b \end{pmatrix}$. (We assume A is $m \times p$, with $n = m + p$, so the LCP is of dimension n as usual.) As the algorithm progresses, we get solutions to parametrized LPs with b replaced by $b - \begin{pmatrix} \delta^m \\ \cdots \\ \delta \end{pmatrix} z_0$, and c replaced by $c + \begin{pmatrix} \delta^n \\ \cdots \\ \delta^{m+1} \end{pmatrix} z_0$. So we can view the algorithm (assuming z_0 decreases steadily) as working on (D), imposing the constraints one by one (with the perturbed $b_i \ll -1$ for all i), and then restoring the true values of the b_i's one by one.

Example:

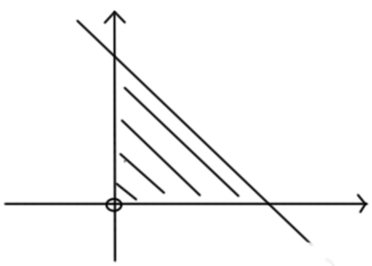
$$(D) \qquad \begin{aligned} \max \quad & y_1 + 3y_2 \\ & y_1 + y_2 \le 4 \\ & -y_1 + 2y_2 \le -1 \\ & y \ge 0. \end{aligned}$$



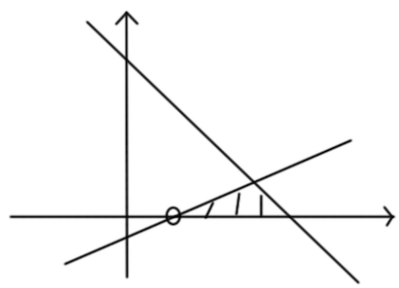
Optimal solution (3;1) for original LP



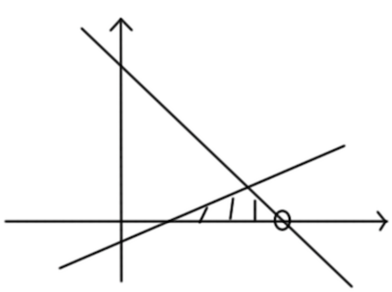
$z_0 = \delta^{-4.5}$, no constraints imposed



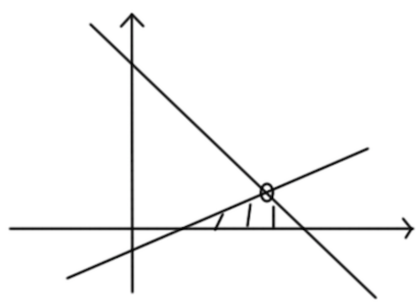
$z_0 = \delta^{-3.5}$, first constraint imposed



$z_0 = \delta^{-2.5}$, both constraints imposed



$z_0 = \delta^{-1.5}$, true b_1 restored



$z_0 = \delta^{-0.5}$, true b_2 restored

This describes the algorithm, but not in a useful way. The column $-d$ associated with z_0 will be basic until the algorithm stops, and having $\begin{pmatrix} -\delta^n \\ \cdots \\ -\delta \end{pmatrix}$, $0 < \delta \ll 1$, where δ is a parameter sufficient close to 0, in the basis matrix is horrible.

Old view:

$$Iw + (-d)z_0 + (-M)z = q. \tag{1}$$

New view: If we have a solution with $z_0 > 0$, we get

$$I(\frac{w}{z_0}) + (-q)(\frac{1}{z_0}) + (-M)(\frac{z}{z_0}) = d \tag{2}$$

and from now on, we let $v = \frac{w}{z_0}$, $y_0 = \frac{1}{z_0}$ and $y = \frac{z}{z_0}$. So now the basis consists of some columns of $[I, -q, -M]$ and d is the RHS. Suppose we have a non-degenerate a.c. but not complementary basic solution to (1). The basic variable will be z_0, w_J for some $J \subseteq N := \{1, 2, \cdots, n\}$, $z_K, K \subseteq N$ with both w_l and z_l nonbasic, and $J \cup K \cup \{l\}$ a disjoint partition of N. Then v_J, y_0 and y_K will be the basic variables in an a.c. basic solution to (2). Recall $d = \begin{pmatrix} \delta^n \\ \cdots \\ \delta \end{pmatrix}$, for $0 < \delta \ll 1$ and this does not lie in any hyperplane formed by $n - 1$ columns of $[I, -q, -M]$, so we get a non-degenerate basic solution. In terms of v, y and (2), the LL algorithm operates as follows:

Algorithm:

- Step 0: If $q \geq 0$, we have a complementary solution: stop. Otherwise, let $B = I$ be the initial basis in (2). Introduce y_0 with column $-q$ into the basis: since q isn't greater than or equal to 0, its increase will be blocked, so some v_l goes to 0. Set $J = N \setminus \{l\}$, $K = \emptyset$.

- Step 1: We have a feasible basis B, J, K and l, giving an a.c. but not complementary solution to (2). Just one of v_l and y_l has just hit zero. Increase its complement. If its increase is blocked by y_0 hitting 0, then by rescaling we have a secondary ray of (1); go to step 2. If its increase is blocked by v_j or y_k, make the pivot, update B, J, K, and l and go to step 1. If its increase is unblocked, we have a ray of solutions:

$$(v_J; y_0; y_K; 0) + \lambda(\bar{v}_J; \bar{y}_0; \bar{y}_K; 1),$$

  where the last component corresponds to the variable we are increasing, v_l or y_l. If $\bar{y}_0$=0, then y_0 (and hence z_0) stays the same, and by scaling, we find a secondary ray of (1); go to step 2. If $\bar{y}_0 > 0$, then by rescaling, we get a complementary solution; go to step 3.

- Step 2(Failure): We have (by scaling) a secondary ray of (1). Stop!

- Step 3(Success): Suppose we were increasing y_l, then ray termination implies

$$I\bar{v} + (-q)\bar{y}_0 - M\bar{y} - m_l \cdot 1 = 0,$$

where $\bar{v}_K = 0$, $\bar{y}_J = 0$. Then $I(\frac{\bar{v}}{\bar{y}_0}) - M(\frac{\bar{y}}{\bar{y}_0} + \frac{1}{\bar{y}_0}e_l) = q$, which gives a complementary solution to (1). Similarly if we were increasing v_l.

Question: When is a particular partition $J \cup K \cup l = N$ feasible?
The basis matrix B has the J columns of I, the column $-q$ and the K columns of $-M$:

$$B = \begin{bmatrix} I_{JJ} & -q_J & -M_{JK} \\ 0 & -q_L & -M_{LK} \end{bmatrix}.$$

with $L := K \cup \{l\}$. For simplicity, let $F := [q, M]$, with columns indexed 0 through n, and let $H := \{0\} \cup K$: then

$$B = \begin{bmatrix} I_{JJ} & -F_{JH} \\ 0 & -F_{LH} \end{bmatrix},$$

with inverse:

$$C := B^{-1} = \begin{bmatrix} I_{JJ} & -F_{JH}F_{LH}^{-1} \\ 0 & -F_{LH}^{-1} \end{bmatrix}.$$

For feasibility, we want $B^{-1}d = C\left(\begin{smallmatrix} \delta^n \\ \cdots \\ \delta \end{smallmatrix}\right) \geq 0$. So feasibility "depends on" the signs of the column of $\left(\begin{smallmatrix} -F_{JH}F_{LH}^{-1} \\ -F_{LH}^{-1} \end{smallmatrix}\right)$ corresponding to the last index in L.