Mathematical Programming II	Lecture 8
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Diameters of Polyhedra

So far we have seen some bad news regarding bounds for diameters of polyhedra:

- The number of vertices can be super-exponential;
- The Hirsch conjecture fails.

In this lecture we will have some good news:

- The Hirsch conjecture holds for some polyhedra;
- There is a subexponential bound on $\Delta(d, n)$ (but not polynomial).

Lemma 1 *Let P be a d-polyhedron with n facets. Choose $0 \neq a \in \mathbb{R}^d$ and $a_0 \in \mathbb{R}$ so that*

$$P \subseteq \{x \in \mathbb{R}^d : a^T x \geq a_0\} \text{ and } P' = \{x \in P : a^T x = a_0\}$$

is nonempty. Then P' is a d'-polyhedron with n' facets, $d' < d$, $n' < n$, and all vectors of P' are vertices of P, with two adjacent in P' if and only if they are adjacent in P.

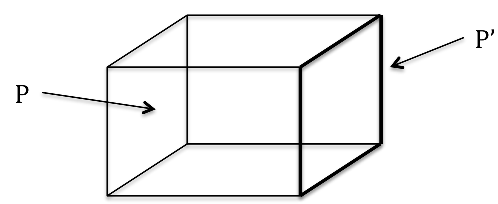

Figure 1: Example of P and P' polyhedra for a 3-dim hyperrectangle.

Proof: Exercise.

Definition 1 *A $(0,1)$-polytope in $\mathbb{R}^d$ is the convex hull of a subset of the $(0,1)$-vectors in $\mathbb{R}^d$.*

Theorem 1 *(D. Naddef, 1989) If P is a $(0,1)$-polytope in $\mathbb{R}^d$ with n facets, then $\delta(P) \leq \min\{d, n-d\}$.*

Proof: (**of Theorem 1**) (i) We need $\delta(P) \leq d$. We proceed by induction: true for $d = 1$. Suppose it is true for dimension less than d, and consider a $(0,1)$-polytope P of dimension d. Let v and w be vertices of P. If $v_i = w_i = 0$ for some i, then choose $a = e_i$ and $a_0 = 0$ in the lemma, and note that v and w are vertices of the $(0,1)$-polytope P' of lower dimension, so $d_P(v, w) \leq d_{P'}(v, w) \leq d' < d$, so we are good. Similarly, if $v_i = w_i = 1$ for some i. So, assume $v = 0$ and $w = e$. Then any edge from v goes to a vertex u of P with some $u_i = 1$. So, $d_P(v,w) \le d_P(v,u) + d_P(u,w) \le 1 + (d-1) = d$.

(*ii*) We now show $\delta(P) \le n - d$ by induction on d.
If v and w both lie on the same facet F, say defined by $a^T x = a_0$, of P, then $a \neq 0$ implies, say, $a_d \neq 0$, and then

$$F = \{x \in P : a^T x = a_0\} = \left\{(x_1; \ldots; x_d) \in P : x_d = \frac{a_0 - a_1 x_1 - \cdots - a_{d-1}x_{d-1}}{a_d}\right\}.$$

Look at $P' = \{\tilde{x} := (x_1; \ldots; x_d) \in \mathbf{R}^{d-1} : \left(x_1; \ldots; x_{d-1}; \frac{a_0 - a_1 x_1 - \cdots - a_{d-1}x_{d-1}}{a_d}\right) \in F\}$, a $(0,1)$-polytope in $\mathbf{R}^{d-1}$ with at most $n-1$ facets: then $d_P(v,w) \le d_{P'}(\tilde{v},\tilde{w}) \le (n-1)-(d-1) = n-d$, by the induction hypothesis.
If v and w do not lie on a common facet, there must be at least $2d$ facets (d for v, d for w) and $d_P(v,w) \le d = 2d - d \le n - d$. □

There is an alternative proof for $\delta(P) \le d$ that goes as follows.
Take any vertices v and w, and without loss of generality assume $v = 0$. Consider minimizing $e^T x$ over P by the simplex method with some anti-cycling rule, starting at w. Since the objective is integer on vertices, with initial value at most d and final value 0, this means at most d nondegenerate steps. □

Now we prove the following theorem. We will use the lemma above several times. Also, we use the easily established fact that $\Delta(d,n)$ is monotonic in n for fixed d.

Theorem 2 *(Basically Kalai-Kleitman) For* $1 \le d \le n$, $\Delta(d,n) \le d^{\log n} = n^{\log d}$.

Here, the logarithms are to base 2. Note that the log of both $d^{\log n}$ and $n^{\log d}$ is $(\log d)(\log n)$, **polynomial** in $\log d$, $\log n$. So this bound is **quasipolynomial**. The proof uses the following lemma.

Lemma 2 *(Kalai-Kleitman) For* $1 \le d \le \lfloor \frac{n}{2} \rfloor$, $\Delta(d,n) \le (d-1, n-1) + 2\Delta(d, \lfloor \frac{n}{2} \rfloor) + 2$.

Proof: **(of Lemma 2)** Choose a d-polyhedron P with n facets and two vertices v and w so that $d_P(v,w) = \Delta(d,n)$. Without loss of generality, we can assume P is *simple*, so that all vertices lie on exactly d facets. If v and w both lie on a common facet P', then $d_P(v,w) \le \Delta(d-1, n-1)$. Suppose not. Let k_v denote the largest k so that there is a set $\mathcal{F}_v$ of at most $\lfloor \frac{n}{2} \rfloor$ facets with all paths from v of length at most k meeting only facets in $\mathcal{F}_v$. This makes sense since paths of length 0 meet only d facets, while paths of length $\delta(P)$ meet all n facets. Define k_w and $\mathcal{F}_w$ similarly.

Claim 1 $k_v \le \Delta(d, \lfloor \frac{n}{2} \rfloor)$.

Proof: **(of Claim 1)** Let P_v denote the d-polyhedron defined by the m $(= |\mathcal{F}_v| \le \lfloor \frac{n}{2} \rfloor)$ inequalities defining the facets in $\mathcal{F}_v$. Choose a shortest path in P from v of length k_v to a vertex of P, say t.

Claim 2 *This is also the shortest path in P_v from v to t.*

Indeed, any shorter path cannot be a path in P, so it would have to meet a facet of P not in $\mathcal{F}_v$. But this is a contradiction.

So $k_v = d_P(v,t) = d_{P_v}(v,t) \le \Delta(d,m) \le \Delta(d, \lfloor \frac{n}{2} \rfloor)$, establishing Claim 1. Similarly, $k_w \le \Delta(d, \lfloor \frac{n}{2} \rfloor)$.

By definition, if we allow ourselves to go at most $k_v + 1$ steps from v, we can reach a set $\mathcal{G}_v$ of facets with $|\mathcal{G}_v| > \lfloor \frac{n}{2} \rfloor$. Similarly, if we allow ourselves to go at most $k_w + 1$ steps from w, we can reach a set $\mathcal{G}_w$ of facets with $|\mathcal{G}_w| > \lfloor \frac{n}{2} \rfloor$. So, there is a facet, say G, in both $\mathcal{G}_v$ and $\mathcal{G}_w$, and a vertex t in G with $d_P(v,t) \le k_v + 1$ and a vertex u in G with $d_P(w,u) \le k_w + 1$. Then,

$$\begin{aligned} d_P(v,w) &\le d_P(v,t) + d_P(t,u) + d_P(w,u) \\ &\le d_P(v,t) + d_G(t,u) + d_P(w,u) \\ &\le k_v + 1 + \Delta(d-1, n-1) + k_w + 1 \\ &\le \Delta(d-1, n-1) + 2\Delta(d, \lfloor \frac{n}{2} \rfloor) + 2. \end{aligned}$$

□

Proof: (**of Theorem 2**) By induction on $d + n$.

- $d = 1: \quad LHS = RHS = 1.$
- $d = 2: \quad LHS = n - 2 < n = RHS.$
- $d = 3$: If $n < 6$, then any two vertices are on a common facet, so $\Delta(3,n) \le \Delta(2, n-1) \le n - 3 < n^{\log 3}$. If $n \ge 6$, by the lemma,

  $$\begin{aligned} \Delta(3,n) &\le \Delta(2, n-1) + 2\Delta(3, \lfloor \frac{n}{2} \rfloor) + 2 \\ &= (n-3) + 2(3^{\log \lfloor \frac{n}{2} \rfloor}) + 2 \\ &\le n - 1 + 2(3^{\log n - 1}) \\ &= n - 1 + \frac{2}{3} \cdot 3^{\log n}, \end{aligned}$$

  so we want $n - 1 \le \frac{1}{3} \cdot n^{\log 3}$. You can check that this is true for $n = 6$. The derivative of its left-hand side is 1, and the derivative of its right-hand side is $\frac{\log 3}{3} \cdot n^{\log 3 - 1} > 1$ for $n \ge 6$, and thus it is true for all $n > 6$.

- $d \ge 4$: If $n < 2d$, any two vertices lie on a common facet, so their distance is at most $\Delta(d-1, n-1)$.
- $d \ge 4$ and $n \ge 2d$: If $n = 8$ (so $d = 4$), two vertices not sharing a facet can be joined by

  $$\underbrace{1}_{a} + \underbrace{\Delta(3,7)}_{b} \le 1 + 3^{\log 7} \le 4^{\log 7} \le 4^{\log 8}$$

  steps, where the term a represents any bounded edge from one vertex and the term b represents the steps from the resulting vertex to the other vertex in some facet.

- $d \geq 4$ and $n \geq 2d$: The only remaining case is $n \geq 9$, so $n-1 \geq 8$, and $\log(n-1) \geq 3$. By Lemma 2,

$$\begin{aligned}
\Delta(d,n) &\leq \Delta(d-1,n-1) + 2\Delta(d,\lfloor\frac{n}{2}\rfloor) + 2 \\
&\leq (d-1)^{\log(n-1)} + 2d^{\log n - 1} + 2 \\
&= \left(\frac{d-1}{d}\right)^{\log(n-1)} d^{\log(n-1)} + \frac{2}{d} \cdot d^{\log n} + 2 \\
&\leq \left(\frac{d-1}{d}\right)^{3} d^{\log n} + \frac{2}{d} \cdot d^{\log n} + 2 \\
&= d^{\log n} - \frac{3}{d} \cdot d^{\log n} + \frac{3}{d^2} \cdot d^{\log n} - \frac{1}{d^3} \cdot d^{\log n} + \frac{2}{d} \cdot d^{\log n} + 2 \\
&= \left(1 - \frac{1}{d} + \frac{3}{d^2} - \frac{1}{d^3}\right) d^{\log n} + 2 \\
&\leq \left(1 - \frac{1}{d} + \frac{3}{4d} - \frac{1}{d^3}\right) d^{\log n} + 2 \\
&= d^{\log n} - \frac{1}{4d} \cdot d^{\log n} - \frac{1}{d^3} \cdot d^{\log n} + 2 \\
&\leq d^{\log n},
\end{aligned}$$

since each of the subtracted terms is at least 1. □