Mathematical Programming II	**Lecture 6**
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$$\begin{aligned} w = \quad & dz_0 + Mz + q && (1) \\ 0 \le \quad & w \perp z \ge 0, \ \ z_0 \ge 0 && (2) \end{aligned}$$

We showed last time:

Theorem 1 *Lemke's Algorithm for a monotone LCP will either produce a complementary solution or show that the LCP is infeasible.*

In particular, if the LCP comes from (QP) and (QD) with H positive semidefinite, $M = \begin{bmatrix} H & -A^T \\ A & 0 \end{bmatrix}$, $q = \begin{bmatrix} c \\ -b \end{bmatrix}$, and then any secondary ray has $\bar{z} = \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} \neq 0$, with either $\bar{x}$ showing (QD) infeasible or $\bar{y}$ showing (QP) infeasible. (See the end of the last notes.)

Definition 1 *M is* ***copositive plus*** *if $z^T Mz \ge 0$ for $z \ge 0$, with equality only if $(M+M^T)z = 0$.*

In fact, our proof of the theorem showed the conclusion holds if M is just copositive plus (Lemke). Also, if M is strictly monotone ($z^T Mz > 0$ if $z \neq 0$), there cannot be any secondary ray, so the algorithm gives a complementary solution. Finally, it can be shown that the algorithm always gives a complementary solution if M is a $P-matrix$.

How are complementary pivot algorithms 1 and 2 related?

In fact, Lemke's algorithm can be reduced to the Lemke-Howson algorithm. We add the constraint:

$$w_0 = -z_0 - e^T z + \omega \ge 0$$

to the system to get the LCP:

$$\begin{bmatrix} w_0 \\ w \end{bmatrix} = \begin{bmatrix} -1 & -e^T \\ d & M \end{bmatrix} \begin{bmatrix} z_0 \\ z \end{bmatrix} + \begin{bmatrix} \omega \\ q \end{bmatrix} \tag{3}$$

$$0 \le \begin{bmatrix} w_0 \\ w \end{bmatrix} \perp \begin{bmatrix} z_0 \\ z \end{bmatrix} \ge 0, \qquad z_0 \ge 0 \tag{4}$$

Choose ω so that $z_0 + e^T z < \omega$ for all basic feasible solutions (BFS) to the system (1). Then the constraint we added just intersects rays going to infinity, and thus makes the unbounded polyhedron bounded. We have a complementary solution to this :

$$((w_0; w) = (0; d\omega + q), (z_0; z) = (\omega; 0))$$

Now start complementary pivot algorithm 1 for this augmented LCP starting from this complementary solution.

E.g., search G_0, so start by increasing w_0, meaning come down the primary ray and then pivot as in complementary pivot algorithm 2. Continue until finding a complementary solution to the original LCP, a **desirable** complementary solution ($w_0 > 0$, $z_0 = 0$) to the augmented LCP, or a **secondary ray**, which leads to an **undesirable** complementary solution ($w_0 = 0$, $z_0 > 0$) to the augmented LCP.

Computational complexity of complementary pivot algorithms

Consider the following LCP:

$$M := M_n := \begin{bmatrix} 1 & & & & 0 \\ 2 & 1 & & & \\ 2 & 2 & . & & \\ . & . & . & . & \\ 2 & 2 & 2 & . & 1 \end{bmatrix} \in \Re^{n \times n}, \quad q := q^{(n)} := \begin{bmatrix} -2^n & & & & \\ -2^n & -2^{n-1} & & & \\ . & . & . & & \\ . & . & . & . & \\ -2^n & -2^{n-1} & -2^{n-2} & . & -2 \end{bmatrix} \in \Re^n.$$

Note: $M + M^T = 2ee^T$, so $z^T(M + M^T)z = 2(e^Tz)^2 \geq 0$. Thus, M is monotone and also a $P - matrix$.

We'll use $d := d^{(n)} := e := e^{(n)} := \begin{bmatrix} 1 \\ . \\ . \\ 1 \end{bmatrix} \in \Re^n$.

Besides, $M_n = \begin{bmatrix} 1 & 0 \\ 2e^{(n-1)} & M_{n-1} \end{bmatrix}$, $q^{(n)} = -2^n e^{(n)} + \begin{bmatrix} 0 \\ q^{(n-1)} \end{bmatrix}$.

Therefore, $w = dz_0 + Mz + q$ becomes

$$\begin{bmatrix} w_1 \\ . \\ . \\ w_n \end{bmatrix} = (z_0 - 2^n)e^{(n)} + \begin{bmatrix} z_1 \\ 2z_1 \\ . \\ 2z_1 \end{bmatrix} + \begin{bmatrix} 0 \\ M_{n-1} \end{bmatrix} \begin{bmatrix} z_2 \\ . \\ . \\ z_n \end{bmatrix} + \begin{bmatrix} 0 \\ q^{(n-1)} \end{bmatrix}. \tag{5}$$

Theorem 2 *Lemke's algorithm applied to this LCP takes $2^n - 1$ pivots starting with $z_0, w_1, ..., w_{n-1}$ basic and ending with $z_1, w_2, ..., w_n$ basic with z_0 decreasing from $2^{n+1} - 2$ to 0 in steps of 2.*

Proof: By induction (check that it's true for $n = 1$).

Assume it is true for dimension at most $n - 1$. Consider M_n and $q^{(n)}$. Start with $z_0 = \max(-\frac{q_i}{1}) = 2^n + 2^{n-1} + ... + 2 = 2^{n+1} - 2$ and $w_n = 0$.

Rewrite (5) as

$$w_1 = z_0' + z_1 \tag{6}$$

$$\begin{bmatrix} w_2 \\ . \\ . \\ w_n \end{bmatrix} = e^{(n-1)}z_0' + 2e^{(n-1)}z_1 + M_{n-1} \begin{bmatrix} z_2 \\ . \\ . \\ z_n \end{bmatrix} + q^{(n-1)} \tag{7}$$

with $z_0' := z_0 - 2^n$.

While z'_0 remains positive, so does w_1, so z_1 stays nonbasic, and by the inductive hypothesis the algorithm will perform **exactly** like Lemke's algorithm on the $(n-1)$-dimensional problem taking $2^{n-1}-1$ pivots to reach the solution with $z'_0 = 0$, so $z_0 = 2^n$ and then w_1 hits 0. When w_1 hits zero, the next step is to increase z_1 and subsequent solutions will have z_1 basic and w_1 nonbasic. So we rewrite the system above, interchanging w_1 and z_1: (5) is also equivalent to :

$$z_1 = z''_0 + w_1 \tag{8}$$

$$\begin{bmatrix} w_2 \\ \cdot \\ \cdot \\ w_n \end{bmatrix} = e^{(n-1)} z''_0 + 2e^{(n-1)} w_1 + M_{n-1} \begin{bmatrix} z_2 \\ \cdot \\ \cdot \\ z_n \end{bmatrix} + q^{(n-1)} \tag{9}$$

with $z''_0 := 2^n - z_0$.

Again by the induction hypothesis there is a path of **a.c.** solutions to the system with z''_0 decreasing from $2^n - 2$ to 0, i.e., z_0 increasing from 2 to 2^n in steps of 2.

Reverse this path and add it to the first:

Table 1: Basic variables change

Step 1	$z_0 > 0$,	w_1,	w_2, ..., w_{n-1}	$z_0 = 2^{n+1} - 2$, $z'_0 = 2^n - 2$
.	.	.		
.	.	w_1		
2^{n-1}	z_0 ,	0,	z_2, ..., w_n	$z_0 = 2^n$, $z'_0 = 0$, $z''_0 = 0$, $(w_1 = 0)$
.	.	z_1		
.	.	.		
$2^n - 1$	z_0 ,	z_1,	w_2, ..., w_{n-1}	$z_0 = 2$, $z''_0 = 2^n - 2$
2^n	compl soln	z_1,	w_2, ..., w_{n-1}, w_n	$z_0 = 0$, $z''_0 = 2^n$ A little way out on the primary ray of the problem above

Note that we have added an extra BFS at the end, corresponding to $z''_0 = 2^n$, so a little way out on the primary ray of this $(n-1)$-dimensional problem, but also corresponding to $z_0 = 0$, so a complementary solution to the original LCP. We also have the correct set of basic variables (we used the inductive hypothesis to make sure the BFS in the middle coincided).

The number of BFS in total is 2^{n-1} (the first set) plus $2^{n-1} - 1$ (the second set, but subtracting one because of the overlap in the middle), plus another one for the last one. Number of BFS: $2^{n-1} + 2^{n-1} - 1 + 1 = 2^n$. z_0 is decreased by 2 in each step.

This concludes the inductive step and thus the proof of the theorem.