Mathematical Programming II	**Lecture 5**
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Consider the last general scheme: perturb the linear system. This is applicable for any LCP.

$$w = dz_0 + Mz + q, \tag{1}$$
$$w \geq 0,\ z_0 \geq 0,\ z \geq 0, \tag{2}$$
$$w_i z_i = 0,\ i = 1, 2, \cdots, n. \tag{3}$$

We take $d > 0$, so that (1)-(2) is feasible (we can choose z_0 sufficiently large).

Assumption 1 *(1)-(2) is nondegenerate, i.e., q cannot be written as a linear combination of fewer than n columns of $[I, -d, -M]$.*

Remark 1 *Two comments:*

1. *We can use lexicographic rules to guarantee this, i.e., replace q by $q + (\epsilon; \epsilon^2; \cdots; \epsilon^n)$ for sufficiently small $\epsilon > 0$.*
2. *In fact, we only need nondegeneracy for solutions with $z_0 > 0$. Equivalently, this is to say that d cannot be expressed as a linear combination of q and fewer than $n - 1$ columns of $[I, -M]$.*

We can view z_0 as a variable, just like the w_i's and z_i's, as the following algorithm does. Alternatively, we may also view z_0 as a **parameter**, so we are solving a sequence of LCPs with q replaced by $q + dz_0$. In particular, for (QP), where $q = (c; -b)$, q is replaced by $(c + d_c z_0; -(b - d_b z_0))$. That is, we perturb the right-hand side of the constraints and the coefficients of the linear part of the objective so that, for sufficiently large z_0, the origin is both feasible and optimal.

Definition 1 *The* ***primary ray*** *for (1) to (3) is the set*

$$\left\{ (w = dz_0 + q; z_0; 0) :\ z_0 \geq \hat{z}_0 := \max\left\{ 0, \max_i \left\{ -\frac{q_i}{d_i} \right\} \right\} \right\}$$
$$= \left\{ (\hat{w}; \hat{z}_0; 0) + \lambda(d; 1; 0) : \lambda \geq 0,\ \hat{w} = d\hat{z}_0 + q \right\}.$$

Definition 2 *The set $\{(w; z_0; z) + \lambda(\bar{w}; \bar{z}_0; \bar{z})\}$ is a* ***secondary ray*** *if it consists of solutions to (1) - (3) (**almost complementary (a.c.)** solutions) with $(\bar{w}; \bar{z}_0; \bar{z}) \neq 0$, where we require*

$$w = dz_0 + Mz + q,\ w \geq 0, z_0 \geq 0, z \geq 0,$$
$$\bar{w} = d\bar{z}_0 + M\bar{z},\ \bar{w} \geq 0, \bar{z}_0 \geq 0, \bar{z} \geq 0,$$
$$w_i z_i = 0, \bar{w}_i z_i = 0, w_i \bar{z}_i = 0, \bar{w}_i \bar{z}_i = 0,\ \forall i,$$
$$z_0 > 0, \bar{z} \neq 0.$$

Note: if $\bar{z} = 0$, then $\bar{w} = d\bar{z}_0$, so we must have $\bar{z}_0 > 0$ and $\bar{w} > 0$. This means that $z = 0$, and this is exactly the primary ray.

Now we can state
Complementary Pivot Algorithm 2 (Lemke 1965).

- **Step 0**: If $q \geq 0$, stop; a complementary solution is $(q; 0)$. Otherwise, set $z_0 := -\frac{q_j}{d_j} := \max_i \left\{-\frac{q_i}{d_i}\right\}$, $z := 0$, $w := dz_0 + q$ as the initial a.c. solution. Note that j is unique due to the **nondegeneracy** assumption. Set z_j as the nonbasic variable to increase.
- **Step 1**: Increase the current nonbasic variable. If its increase is unblocked, go to **Step 2**. Otherwise, pivot to the new a.c. basic feasible solution. If z_0 has hit zero, go to **Step 3**. If w_i (or z_i) has hit zero, ($i \in \{1, \cdots, n\}$), then choose its complement z_i (or w_i) as the new nonbasic variable to increase, and go to **Step 1**.
- **Step 2**: (Failure) STOP. We have found a secondary ray.
- **Step 3**: (Success) STOP. The current basic feasible solution gives a complementary solution to the original LCP.

Remark 2 *By our nondegeneracy assumption, this algorithm is well-defined.*

To analyze this algorithm, consider the graph G, whose nodes are all a.c. basic feasible solutions, with two adjacent if their midpoint is also a.c.

Theorem 1 *All nodes of G have degree 0, 1 or 2, so G is a disjoint union of isolated nodes, paths and cycles. Here degree 0, 1 and 2 nodes are characterized as follows.*

1. *A node of degree 0 is either a complementary solution ($z_0 = 0$), where the increase of z_0 is unblocked; or an a.c. but not complementary solution, where the increases of both of w_i and z_i ($w_i = z_i = 0$) are unblocked.*
2. *A node of degree 1 is either a complementary solution where the increase of z_0 is blocked, or an a.c. but not complementary solution where the increase of **exactly one** of w_i, z_i is blocked.*
3. *A node of degree 2 is an a.c. but not complementary solution, where the increases of both w_i and z_i are blocked.*

Proof: If a node is a complementary solution, $z_0 = 0$, and by nondegeneracy, there are n positive variables, exactly one in each pair of w_i and z_i. Hence the only way to possibly find an adjacent node is to increase z_0, whose increase is either unblocked (degree 0) or blocked (degree 1).

If a node is a.c. but not complementary, then z_0 is positive and there are $n-1$ other variables positive by nondegeneracy. So exactly one pair of variables, (w_i, z_i) are both zero. So our only choice is to increase either w_i or z_i, each of which can be either unblocked or blocked, leading to the statement of the theorem.

Corollary 1 *Lemke's algorithm terminates in a finite number of steps with either a complementary solution or a secondary ray.*

Proof: The algorithm traces a path in G from the endpoint of the primary ray. If this endpoint has degree zero, then the increase of z_j is unblocked, so we have found a secondary ray with $\bar{z}_j > 0$. Otherwise, we trace the path to another node of degree 1.

If this is a complementary solution, we are done. If not, it's an a.c. but not complementary solution, where the increase of exactly one of w_i and z_i is unblocked.

This must correspond to a **ray** of a.c. solutions from the current a.c. solution. This cannot be a ray with $\bar{z} = 0$, because this implies $\bar{w} > 0$, which implies $z = 0$, and we have the initial solution, a contradiction.

So in either case, we have a secondary ray. This finishes the proof of the corollary.

Example 1 *Consider the following LCP, with*

$$q = \begin{pmatrix} -4 \\ -6 \end{pmatrix}, \quad M = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \quad d = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Note*: M is monotone and a P-matrix.*

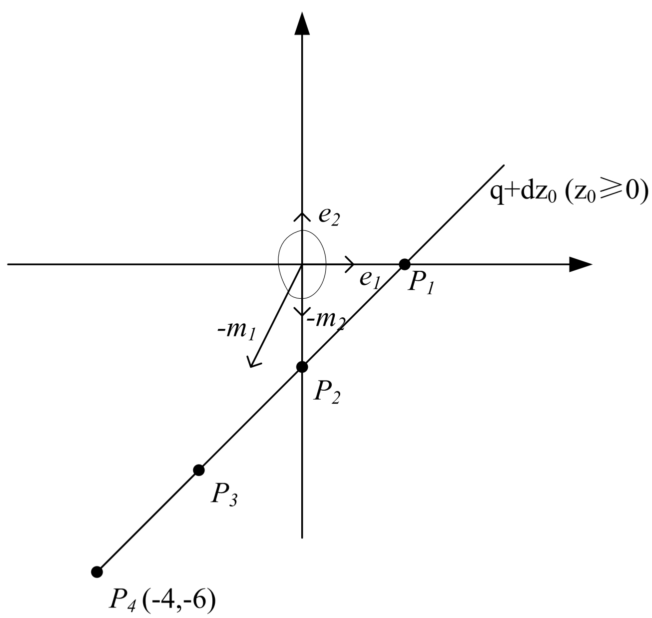

Figure 1: Example 1

In this example, our pivots go from P_1 to P_4. When we reach P_4 we have our desired complementary solution, $(w = (0;2), z = (4;0))$.

1. *At the beginning, choose $\hat{z}_0 = 6$, and we get the point P_1. The corresponding basic variable to increase is z_2.*

2. *Increase* z_2 *to 2, and we reach* P_2*. By this increase,* w_1 *hits zero, and* $z_0 = 4$*. We then choose* z_1 *as the new nonbasic variable to increase.*

3. *Increase* z_1 *to 2, and we reach* P_3*. Then* z_2 *hits zero, and* $z_0 = 2$*. We choose* w_2 *as the new nonbasic variable to increase.*

4. *Increase* w_2 *to 2, and meanwhile* z_1 *goes to 4. Then* $z_0 = 0$*, and we obtain the complementary solution.*

In total, we make $2^2 - 1 = 3$ *pivots to reach the complementary solution.*

Example 2 *Consider the following LCP, with*

$$q = \begin{pmatrix} -3 \\ -1 \end{pmatrix}, \quad M = \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix}, \quad d = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

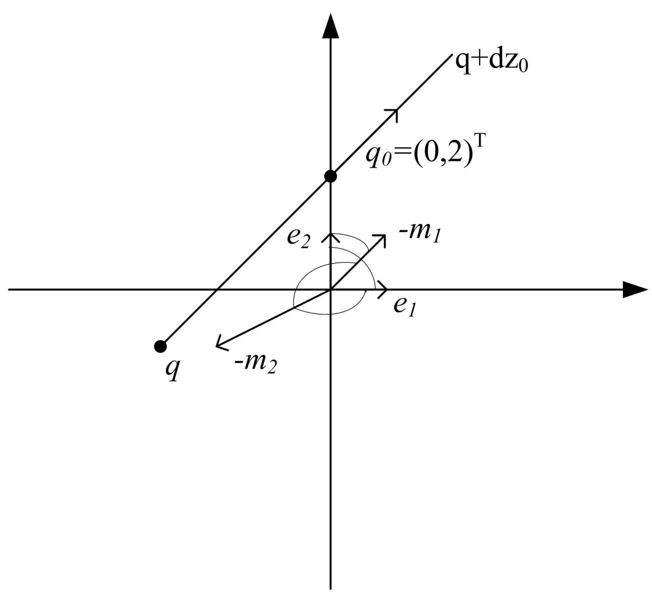

Figure 2: Example 2

We take $\hat{z}_0 = 3$*, and then choose* z_1 *as the nonbasic variable to increase. However, the increase in* z_1 *is unblocked, and we have found a secondary ray, given by*

$$\left\{ \begin{pmatrix} w_1 \\ w_2 \\ z_0 \\ z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 3 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \lambda : \ \lambda \geq 0 \right\}.$$

This results in a failure (even though there is a complementary solution).

Theorem 2 *If M is monotone, failure in a secondary ray implies that the original LCP was* ***infeasible****.*

Proof: We have $(w; z_0; z)$ and $(\bar{w}; \bar{z}_0; \bar{z})$ in the secondary ray, with

a) $\bar{z} \neq 0$.

b) $\bar{z}_0 = 0$ and $\bar{z}^T M \bar{z} = 0$. This is because

$$\begin{aligned} \bar{w} =& d\bar{z}_0 + M\bar{z}, \\ \text{so } 0 = \bar{z}^T \bar{w} =& (\bar{z}^T d)\bar{z}_0 + \bar{z}^T M \bar{z}. \end{aligned}$$

Since $d > 0$, $\bar{z} \neq 0$, $\bar{z} \geq 0$, $\bar{z}_0 \geq 0$ and M is monotone, we have $\bar{z}^T d > 0$, $(\bar{z}^T d)\bar{z}_0 = 0$ and $\bar{z}^T M \bar{z} = 0$, so we must have $\bar{z}_0 = 0$ and $\bar{z}^T M \bar{z} = 0$.

c) $M^T \bar{z} = -\bar{w} \leq 0$.

From (b), we know that $M\bar{z} = \bar{w} \geq 0$. Also, $\bar{z}^T M \bar{z} = 0$ implies $\bar{z}^T (M + M^T)\bar{z} = 0$, so $(M + M^T)\bar{z} = 0$. This indicates that $M^T \bar{z} = -\bar{w} \leq 0$.

d) $\bar{z}^T q < 0$.

To see this we know that

$$0 = \bar{z}^T w = (\bar{z}^T d) z_0 + \bar{z}^T M z + \bar{z}^T q.$$

Here $d > 0$, $\bar{z} \geq 0$ and $\bar{z} \neq 0$, so $\bar{z}^T d > 0$. By the assumption of a secondary ray, $z_0 > 0$, hence $(\bar{z}^T d) z_0 > 0$. Also, $\bar{z}^T M z = (M^T \bar{z})^T z = -\bar{w}^T z = 0$. Altogether this indicates that $\bar{z}^T q < 0$.

So if $(\tilde{w}, \tilde{z})$ were feasible for the original LCP, we would have $\tilde{w} = M\tilde{z} + q$, as well as the inequality

$$0 \leq \bar{z}^T \tilde{w} = \bar{z}^T M \tilde{z} + \bar{z}^T q \tag{4}$$

Since $\bar{z}^T M \tilde{z} = (M^T \bar{z})^T \tilde{z}$, $M^T \bar{z} \leq 0$ and $\tilde{z} \geq 0$, we have $\bar{z}^T M \tilde{z} \leq 0$. However, $\bar{z}^T q < 0$, resulting in a contradiction in (4). Hence we cannot have a feasible solution to the original LCP.

Corollary 2 *If the LCP(M, q) arises from (QP) and (QD), with H positive semidefinite, then termination in a secondary ray gives a certificate for either primal or dual infeasibility.*

Proof: We split $\bar{z}$ as above into $(\bar{x}; \bar{y})$. From (b) we have $\bar{x}^T H \bar{x} = \bar{z}^T M \bar{z} = 0$. Hence $H\bar{x} = 0$. Then from (c) we have $A^T \bar{y} = H\bar{x} + A^T \bar{y} \leq 0$ and $-A\bar{x} \leq 0$. Finally, from (d) we have $c^T \bar{x} - b^T \bar{y} < 0$, so either $c^T \bar{x} < 0$ or $b^T \bar{y} > 0$ (or both). In the first case, $c^T \bar{x} < 0$, $H\bar{x} = 0$, and $A\bar{x} \geq 0$ give a certificate that (QD) is infeasible. In the second, $b^T \bar{y} > 0$ and $A^T \bar{y} \leq 0$ give a certificate that (QP) is infeasible.