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Complementary Pivot Algorithms for the LCP:

$$w = Mz + q \tag{1}$$
$$w \geq 0, z \geq 0 \tag{2}$$
$$w_i z_i = 0, \quad i = 1, 2 \cdots n. \tag{3}$$

Assuming (1)-(2) is nondegenerate, then any solution to (1)-(2) has at least n nonzeroes, and satisfying (3) means at most n nonzeroes. So there are **exactly** n non-zeroes, and such complementary solutions are isolated.

We need to relax one or more of these conditions to allow "freedom to move" in an algorithm.

a) Relax all the complementarity conditions to $w_i z_i = \mu$ for all n and consider solutions as μ decreases to 0. (Interior-point methods, see the paper by Kojima et al. on the homepage).

b) Relax just one of the complementary conditions (today's lecture).

c) Relax the linear system $w = Mz + q$ (next lecture).

Assumption 1 *Equation (1) is non-degenerate, that is q cannot be written as a linear combination of fewer than n columns of $[I, -M]$.*

Remark 1 *We can replace q by $q + (\epsilon; \epsilon^2; \cdots; \epsilon^n)$ for sufficiently small $\epsilon > 0$ to ensure the assumption.*

Let's consider b) first.

Example 1 *(Bimatrix Game) consider the bimatrix game given by the following A and B:*

$$A = \begin{bmatrix} 0 & 0 & 10 \\ 5 & 0 & 0 \\ 0 & 5 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 10 \\ 0 & 3 & 0 \\ 6 & 0 & 0 \end{bmatrix}.$$

The corresponding LCP is:

$$s + \begin{bmatrix} 0 & 0 & 10 \\ 5 & 0 & 0 \\ 0 & 5 & 0 \end{bmatrix} v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \tag{4}$$
$$t + \begin{bmatrix} 0 & 0 & 6 \\ 0 & 3 & 0 \\ 10 & 0 & 0 \end{bmatrix} u = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \tag{5}$$
$$s \geq 0, v \geq 0, t \geq 0, u \geq 0, \qquad s \perp u, t \perp v. \tag{6}$$

Note that $u = 0$, $v = 0$, and $s = t = e$ is one complementary solution. And we want to find another one.

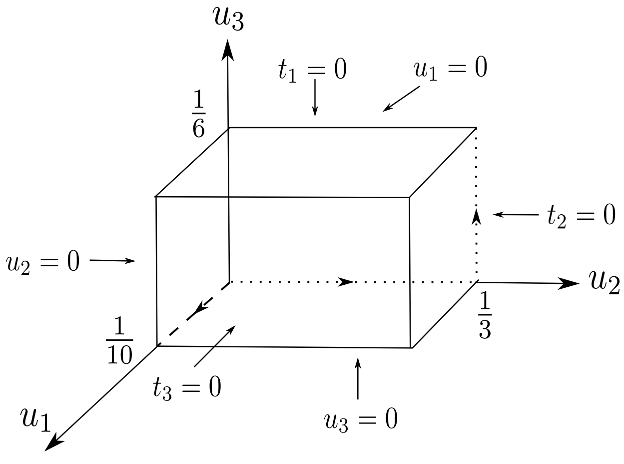

Figure 1: feasible region of u

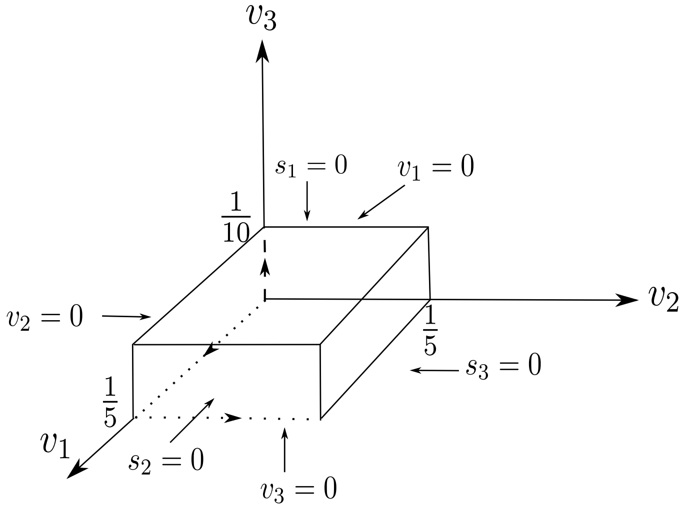

Figure 2: feasible region of v

i) Relax $s_1 u_1 = 0$.

From the trivial solution, we increase u_1, *and* t_3 *hits zero. Then we increase* v_3 *which is the* ***only*** *choice, and* s_1 *hits zero! (Dashed paths in Figures 1 and 2).*

Thus we have a new complementary solution $\bar{\bar{u}} = (\frac{1}{10}; 0; 0)$, $\bar{\bar{v}} = (0; 0; \frac{1}{10})$, *corresponding to the Nash equilibrium:* $\bar{\bar{x}} = (1; 0; 0)$, $\bar{\bar{y}} = (0; 0; 1)$.

ii) Relax $t_1 v_1 = 0$.

From the trivial solution, we increase v_1, *and* s_2 *hits zero. Then we increase* u_2, *and* t_2 *hits zero. We further increase* v_2, *and* s_3 *hits zero. We finally increase* u_3, *and* t_1 *hits zero. (Dotted paths in Figures 1 and 2.)*

Thus we obtain another complementary solution $\hat{u} = (0; \frac{1}{3}; \frac{1}{6})$, $\hat{v} = (\frac{1}{5}; \frac{1}{5}; 0)$. *So the Nash equilibrium is* $\hat{x} = (0; \frac{2}{3}; \frac{1}{3})$, $\hat{y} = (\frac{1}{2}; \frac{1}{2}; 0)$.

In general, suppose we have an LCP(M, q), and we know one complementary solution, say

$(\bar{w}, \bar{z})$. We want to find another complementary solution. We need the following definition.

Definition 1 *For any k=1,2...n, a* ***k-almost complementary*** *(k-a.c.) solution is one satisfying (1)-(2), and $w_i z_i = 0$ for all $i \neq k$. We further define a graph $G = G_k$ with as nodes all k-a.c. basic feasible solutions to (1)-(2). Two such are* ***adjacent*** *if their mid point is also k-a.c.*

Suppose (w, z) is a complementary solution; then for each i, exactly one of w_i, z_i is zero. The only way to find an adjacent node is to increase whichever of w_k and z_k is zero. We either go to infinity on a ray (failure) or reach another k-a.c. basic solution. We say the increase is **unblocked** (respectively **blocked**). Thus the degree of (w, z) is: $\begin{cases} 0, & \text{if increase is unblocked,} \\ 1, & \text{if increase is blocked.} \end{cases}$

Now suppose (w, z) is a k-a.c. basic feasible solution, but is not complementary, so w_k **and** z_k are positive. For each i, at most one of w_i, z_i is positive, and there are exactly n positive variables. So there is exactly one j with both w_j and z_j zero. We have **exactly** two choices of variable to increase to try to find another k-a.c. basic feasible solution: w_j and z_j. The degree of (w, z) is: $\begin{cases} 0, & \text{if increase of both } w_j \text{ and } z_j \text{ is unblocked,} \\ 1, & \text{if increase of exactly one of } w_j,\ z_j \text{ is blocked,} \\ 2, & \text{if increase of both } w_j \text{ and } z_j \text{ is blocked.} \end{cases}$

Theorem 1 *G has nodes of degree only 0, 1 and 2. So it is a disjoint union of isolated nodes, paths and cycles.*

Degree 0 nodes:	*complementary solution*	*where increase of w_k or z_k is unblocked.*
	non-complementary solutions	*where increase of both w_k and z_k is unblocked.*
Degree 1 nodes:	*complementary solution*	*where increase of w_k or z_k is blocked.*
	non-complementary solutions	*where increase of exactly one of w_k and z_k is blocked.*
Degree 2 nodes:	*non-complementary solutions*	*where increase of both w_k and z_k is blocked.*

Corollary 1 *If the set of feasible solutions of LCP(M, q) is bounded, then there is an even number of complementary solutions.*

Remark 2 *They are the end points of the paths in G.*

Corollary 2 *Every bimatrix game has a Nash equilibrium.*

Remark 3 *In fact, the number of Nash equilibria is an odd number with Assumption 1.*

This leads to our first algorithm.

Complementary Pivot Algorithm 1 (Lemke-Howson, 1964).

Suppose we have one complementary solution $(\bar{w}, \bar{z})$ to LCP(M, q), and we want to find another complementary solution (assuming nondegeneracy).

Step 0. Set (w, z) to $(\bar{w}, \bar{z})$. Choose $1 \leq k \leq n$. Increase whichever of (w_k, z_k) is zero.

Step 1. If the increase of the current variable is unblocked, then go to Step 2. Otherwise, make the pivot so that some w_j or z_j hits zero. If $j = k$, go to Step 3. Otherwise, choose as

the new variable to increase the complement z_j (or w_j) of the variable that just hit zero. Go to Step 1.
Step 2. (Failure) We have found an unbounded ray of k-a.c. solutions: Stop.
Step 3. (Success) Our solution $(\bar{\bar{w}}, \bar{\bar{z}}) = (w, z)$ is complementary.

Theorem 2 *Complementary Pivot Algorithm 1 always terminates in a finite number of steps.*

We can search **all** G_k's to try to find more complementary solution. But we could have the following situation: