In this lecture, we will consider a topic in Game Theory: bimatrix games and their relation to LCPs. We start with a definition.

**Definition 1** *(Bimatrix Games) A bimatrix game is given by two $m \times n$ matrices $A$ and $B$. Player I chooses $i \in \{1, \ldots, m\}$ and Player II chooses $j \in \{1, \ldots, n\}$. The players then get payoffs $a_{ij}$ and $b_{ij}$ respectively.*

We consider noncooperative theory: so we assume that the players do not (or cannot) cooperate in choosing their strategies.

**Definition 2** *(Nash Equilibrium) A pair $(\bar{i}, \bar{j})$ is a (pure strategy) Nash equilibrium if $a_{i\bar{j}} \leq a_{\bar{i}\bar{j}}$ for all $i$ and $b_{\bar{i}j} \leq b_{\bar{i}\bar{j}}$ for all $j$, i.e., there are no incentives for a unilateral switch.*

To get an idea of Nash equilibrium, here are some examples.

**Example 1** *The Prisoner's Dilemma: Consider two players who are accused of a crime and are detained and questioned separately by the police. They have two options, either staying quiet, which we will denote as Q, or confessing, which we will denote as C. The resulting payoffs to the two players are given below, and correspond to the (negative of the) years to be served in prison in each scenario.*

|  | $Q$ | $C$ |
| --- | --- | --- |
| $Q$ | $(-2, -2)$ | $(-10, -1)$ |
| $C$ | $(-1, -10)$ | $(-5, -5)$ |

*Here, the strategy pair $(2, 2)$, giving rise to the payoff $(-5, -5)$, is the unique Nash equilibrium, which corresponds to both players confessing. Starting from the Nash equilibrium, if Player I makes a switch, he/she will have to serve 10 years rather than 5 years. This is the same for Player II. Note that both players are better off if they both stay quiet, but this requires cooperation (and enforcement of the agreement).*

**Example 2** *The Honeymoon Problem: Suppose Alice and Bob are on their Honeymoon and Alice wants to see the ballet (B) but Bob wants to see a baseball game (G). The payoffs of their actions are shown below.*

|  | $B$ | $G$ |
| --- | --- | --- |
| $B$ | $(1, 5)$ | $(0, 0)$ |
| $G$ | $(0, 0)$ | $(5, 1)$ |

*In this case, both $(1, 1)$ and $(2, 2)$ are Nash equilibria. This is the same even if we have payoffs $(1, 1)$ and $(5, 5)$ instead of $(1, 5)$ and $(5, 1)$. Note that NE are not unique, and one might dominate another!*

**Example 3** *Dime Matching: Suppose Alice and Bob are going to play a dime matching game. Each player chooses heads (H) or tails (T). The payoff of the actions are shown below. This is a zero-sum game.*

$$
\begin{array}{c c c}
 & H & T \\
H & (5, -5) & (-5, 5) \\
T & (-5, 5) & (5, -5)
\end{array}
$$

*In this case, there is* no *pure-strategy NE. At any strategy pair, one player has an incentive to change his/her decision.*

In general, we want to consider *mixed* strategies. Let $X := \left\{ x : e_m^T x = 1, x \geq 0 \right\}$ and $Y := \left\{ y : e_n^T y = 1, y \geq 0 \right\}$, where $e_k := (1; \ldots ; 1) \in \mathbf{R}^k$.

If Player I chooses $x$ (plays pure strategy $i$ with probability $x_i$) and Player II chooses $y$ (plays pure strategy $j$ with probability $y_j$), then I gets expected payoff $x^T A y$ and II gets expected payoff $x^T B y$. We assume that players want to maximize these. Let us define the NE for this case when the players are using mixed strategies.

**Definition 3** *(Mixed Strategy Nash Equilibrium)* $(\bar{x}, \bar{y}) \in X \times Y$ *is a (mixed strategy) Nash equilibrium if* $x^T A \bar{y} \leq \bar{x}^T A \bar{y}$ *and* $\bar{x}^T B y \leq \bar{x}^T B \bar{y}$ *for all* $x \in X$ *and* $y \in Y$ *respectively.*

This has bilinearity, an infinite number of constraints and no complementarity conditions. But note that the condition $x^T A \bar{y} \leq \bar{x}^T A \bar{y}$ for all $x \in X$ holds if and only if $A \bar{y} \leq \left( \bar{x}^T A \bar{y} \right) e_m$ which takes care of the infinite number of constraints. This is also equivalent to

$$
A\bar{y} \leq \alpha e_m, \quad \alpha = \bar{x}^T A \bar{y}, \text{ or}
$$
$$
A\bar{y} \leq \alpha e_m, \quad \text{for some } \alpha \in \mathbf{R} \text{ with equality in the } i\text{th position if } x_i > 0.
$$

Similarly, $\bar{x}^T B y \leq \bar{x}^T B \bar{y}$ for all $y \in Y$ holds if and only if $B^T \bar{x} \leq \beta e_n$ for some $\beta \in \mathbf{R}$ with equality in the $j$th position if $\bar{y}_j > 0$. Note that the components of $\bar{x}$ and $\bar{y}$ depend on the *other* player's payoffs.

We would like $\alpha, \beta > 0$ so we can scale $\bar{x}$ and $\bar{y}$ to get $e_m$ and $e_n$ on the RHS. To do so, we need to perturb the bimatrix game.

**Proposition 1** *For any* $\gamma \in \mathbf{R}$, $\delta \in \mathbf{R}$, $(\bar{x}, \bar{y})$ *is a NE for* $(A, B)$ *if and only if it is a NE for* $\left( A + \gamma e_m e_n^T, B + \delta e_m e_n^T \right)$.

**Proof:** Exercise.

With this proposition, we can assume without loss of generality that $A, B > 0$ entry-wise. Then, $\alpha$ and $\beta$ must be positive, so we can scale by $\alpha$ and $\beta$ to get

$$
\begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 0 & -A \\ -B^T & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} e_m \\ e_n \end{bmatrix}
$$

and $0 \leq [s; t] \perp [u; v] \geq 0$ so we have an LCP!

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**Theorem 1** *(Relation of NE and LCP) Assume $A, B > 0$. Then if $(\bar{x}, \bar{y})$ is a NE of $(A, B)$, $\left(\bar{u} = \bar{x}/\bar{x}^T B\bar{y}, \bar{v} = \bar{y}/\bar{x}^T A\bar{y}\right)$ with suitable $(\bar{s}, \bar{t})$ is a complementary solution to the LCP above.*

*Conversely, if $(\bar{s}, \bar{t}, \bar{u}, \bar{v})$ is a complementary solution to the LCP above, and $(\bar{u}, \bar{v}) \neq (0, 0)$, then $\left(\bar{x} = \bar{u}/e_m^T \bar{u}, \bar{y} = \bar{v}/e_n^T \bar{v}\right)$ is a NE for $(A, B)$.*

**Proof:** We have already shown the first part.

For the converse, note that if $\bar{u} \neq 0$, some $\bar{u}_i > 0$ so some $\bar{s}_i = 0$, so $\bar{v} \neq 0$. Similarly, $\bar{v} \neq 0$ implies $\bar{u} \neq 0$. So $\bar{x}$ and $\bar{y}$ are well defined in $X \times Y$. But, $A\bar{v} \leq e_m$, with equality in the $i$th position if $\bar{u}_i > 0$, implies that $A\bar{y} \leq \alpha e_m$, with $\alpha = 1/e_n^T \bar{v}$, with equality in the $i$th position if $x_i > 0$.

Similarly, $B^T \bar{x} \leq \beta e_n$ for $\beta = 1/e_m^T \bar{u}$ with equality in the $j$th position if $\bar{y}_j > 0$. Hence, $(\bar{x}, \bar{y})$ is a NE for $(A, B)$.

Note that $M = \begin{bmatrix} 0 & -A \\ -B^T & 0 \end{bmatrix}$ is neither a $P$-matrix nor (in general) monotone. It is monotone if $B = -A$, which corresponds to a zero-sum game. Note that the set of *feasible* solutions to this LCP is bounded.

For other applications of LCP, see the paper of Ferris and Pang on the course homepage. Note that the complementary variables in equilibrium problems are the prices of goods and their corresponding excess supplies. Also, see the slides by Mihai Anitescu on the homepage. Here, complementary variables in mechanics problems model non-intersection of bodies (the distance between them is nonnegative) and the corresponding normal force.

And for an encore:

OR 6310: Mathematical Programming II. Spring 2014.

Homework Set 1. Due: Thursday February 20.

1. (KKT solutions and minimizers)

a) Find all KKT solutions ($\bar{x}$'s which with suitable multipliers satisfy the KKT conditions) for $\min\{x_1 - x_1^2 + x_2^2 : -1 \leq x_1 \leq 1\}$, and characterize which are global minimizers, which local but not global minimizers, and which are not even local minimizers.

b) By making a small change to the problem in (a), find a quadratic programming problem where the best KKT solution is not the global minimizer. [This shows that, even if you could "solve" arbitrary LCPs, you can't guarantee finding global minimizers for QPs.]

2. (Symmetric quadratic programming duality)

a) Consider the unconstrained quadratic minimization problem

$$(P): \quad \min_v f(v) := d^T v + \frac{1}{2} v^T H v,$$

where $H = H^T \in \Re^{n \times n}$ is positive semidefinite. Suppose $\nabla f(\bar{v}) = d + H\bar{v} = 0$. Show directly that $\bar{v}$ is a global minimizer for $(P)$, and hence that any two solutions $\bar{v}$ have the same value of $f$. Show also that, if $d + Hv = 0$ has no solution, then $(P)$ is unbounded below.

b) Now consider the constrained quadratic programming problem

$$(QP): \quad \min_{x,u} c^T x + \frac{1}{2} x^T H x + \frac{1}{2} u^T G u, \quad Ax + Gu \geq b, \ x \geq 0.$$

Here $A, H, b, c$ are as in class and $G = G^T \in \Re^{m \times m}$. Note that if $G = 0$, this is the problem considered in class. Show that $(QP)$ is equivalent to the min-max problem $\min_{x,u} \max_{y \geq 0, s \geq 0} L(x, u, y, s)$, where $L(x, u, y, s)$ is the Lagrangian function

$$L(x, u, y, s) := c^T x + \frac{1}{2} x^T H x + \frac{1}{2} u^T G u + (b - Ax - Gu)^T y + (-x)^T s.$$

Henceforth assume that $H$ and $G$ are positive semidefinite. Next show that the max-min problem $\max_{y \geq 0, s \geq 0} \min_{x,u} L(x, u, y, s)$ is equivalent to the dual problem below (note that this coincides with the dual problem $(QD)$ stated in class if $G = 0$):

$$(QD): \quad \max_{y,v} b^T y - \frac{1}{2} y^T G y - \frac{1}{2} v^T H v, \quad A^T y - Hv \leq c, \ y \geq 0.$$

c) Hence show weak duality directly for this pair of problems.

d) By writing $(QD)$ in the form of $(QP)$, show that the "dual of the dual is the primal."

3. Our formulation of finding Nash equilibria in a bimatrix game as an LCP does not distinguish one Nash equilibrium from another. Find an LCP so that any nontrivial complementary solution gives a Nash equilibrium where I's expected payoff is at least $\alpha$ and II's expected payoff is at least $\beta$. Is it easy to find such Nash equilibria by the same algorithm as discussed in class?

4. We made sure that $A$ and $B$ had all positive entries, and then set up a bounded linear system of equations and inequalities to find Nash equilibria of the bimatrix game $(A, B)$.

a) Suppose instead we start by ensuring that all entries of $A$ and $B$ are *negative*, and then consider the LCP defined by

$$M = \begin{pmatrix} 0 & -A \\ -B^T & 0 \end{pmatrix}, \quad q = \begin{pmatrix} -e_m \\ -e_n \end{pmatrix}.$$

Is the corresponding polyhedron bounded? Prove a theorem relating Nash equilibria of the bimatrix game to complementary solutions of this LCP.

b) Try to modify the algorithm we discussed, using $k$-a.c. basic feasible solutions, to attack the LCP in (a). Show how to initialize it (a couple of special pivots may be required). Do not worry about secondary rays.