Mathematical Programming II	**Lecture 3**
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In this lecture, we will consider a topic in Game Theory: bimatrix games and their relation to LCPs. We start with a definition.

Definition 1 *(Bimatrix Games) A bimatrix game is given by two $m \times n$ matrices A and B. Player I chooses $i \in \{1, \ldots, m\}$ and Player II chooses $j \in \{1, \ldots, n\}$. The players then get payoffs a_{ij} and b_{ij} respectively.*

We consider noncooperative theory: so we assume that the players do not (or cannot) cooperate in choosing their strategies.

Definition 2 *(Nash Equilibrium) A pair $(\bar{i}, \bar{j})$ is a (pure strategy) Nash equilibrium if $a_{i\bar{j}} \leq a_{\bar{i}\bar{j}}$ for all i and $b_{\bar{i}j} \leq b_{\bar{i}\bar{j}}$ for all j, i.e., there are no incentives for a unilateral switch.*

To get an idea of Nash equilibrium, here are some examples.

Example 1 *The Prisoner's Dilemma: Consider two players who are accused of a crime and are detained and questioned separately by the police. They have two options, either staying quiet, which we will denote as Q, or confessing, which we will denote as C. The resulting payoffs to the two players are given below, and correspond to the (negative of the) years to be served in prison in each scenario.*

	Q	C
Q	$(-2, -2)$	$(-10, -1)$
C	$(-1, -10)$	$(-5, -5)$

Here, the strategy pair $(2, 2)$, giving rise to the payoff $(-5, -5)$, is the unique Nash equilibrium, which corresponds to both players confessing. Starting from the Nash equilibrium, if Player I makes a switch, he/she will have to serve 10 years rather than 5 years. This is the same for Player II. Note that both players are better off if they both stay quiet, but this requires cooperation (and enforcement of the agreement).

Example 2 *The Honeymoon Problem: Suppose Alice and Bob are on their Honeymoon and Alice wants to see the ballet (B) but Bob wants to see a baseball game (G). The payoffs of their actions are shown below.*

	B	G
B	$(1, 5)$	$(0, 0)$
G	$(0, 0)$	$(5, 1)$

In this case, both $(1, 1)$ and $(2, 2)$ are Nash equilibria. This is the same even if we have payoffs $(1, 1)$ and $(5, 5)$ instead of $(1, 5)$ and $(5, 1)$. Note that NE are not unique, and one might dominate another!

Example 3 *Dime Matching: Suppose Alice and Bob are going to play a dime matching game. Each player chooses heads (H) or tails (T). The payoff of the actions are shown below. This is a zero-sum game.*

	H	T
H	$(5,-5)$	$(-5,5)$
T	$(-5,5)$	$(5,-5)$

In this case, there is no *pure-strategy NE. At any strategy pair, one player has an incentive to change his/her decision.*

In general, we want to consider *mixed* strategies. Let $X := \left\{x : e_m^T x = 1, x \geq 0\right\}$ and $Y := \left\{y : e_n^T y = 1, y \geq 0\right\}$, where $e_k := (1; \ldots ; 1) \in \mathbf{R}^k$.

If Player I chooses x (plays pure strategy i with probability x_i) and Player II chooses y (plays pure strategy j with probability y_j), then I gets expected payoff $x^T Ay$ and II gets expected payoff $x^T By$. We assume that players want to maximize these. Let us define the NE for this case when the players are using mixed strategies.

Definition 3 *(Mixed Strategy Nash Equilibrium)* $(\bar{x}, \bar{y}) \in X \times Y$ *is a (mixed strategy) Nash equilibrium if* $x^T A\bar{y} \leq \bar{x}^T A\bar{y}$ *and* $\bar{x}^T By \leq \bar{x}^T B\bar{y}$ *for all* $x \in X$ *and* $y \in Y$ *respectively.*

This has bilinearity, an infinite number of constraints and no complementarity conditions. But note that the condition $x^T A\bar{y} \leq \bar{x}^T A\bar{y}$ for all $x \in X$ holds if and only if $A\bar{y} \leq \left(\bar{x}^T A\bar{y}\right) e_m$ which takes care of the infinite number of constraints. This is also equivalent to

$$
\begin{aligned}
&A\bar{y} \leq \alpha e_m, \quad \alpha = \bar{x}^T A\bar{y}, \text{ or} \\
&A\bar{y} \leq \alpha e_m, \quad \text{for some } \alpha \in \mathbf{R} \text{ with equality in the } i\text{th position if } x_i > 0.
\end{aligned}
$$

Similarly, $\bar{x}^T By \leq \bar{x}^T B\bar{y}$ for all $y \in Y$ holds if and only if $B^T\bar{x} \leq \beta e_n$ for some $\beta \in \mathbf{R}$ with equality in the jth position if $\bar{y}_j > 0$. Note that the components of $\bar{x}$ and $\bar{y}$ depend on the *other* player's payoffs.

We would like $\alpha, \beta > 0$ so we can scale $\bar{x}$ and $\bar{y}$ to get e_m and e_n on the RHS. To do so, we need to perturb the bimatrix game.

Proposition 1 *For any* $\gamma \in \mathbf{R}$, $\delta \in \mathbf{R}$, $(\bar{x}, \bar{y})$ *is a NE for* (A, B) *if and only if it is a NE for* $\left(A + \gamma e_m e_n^T, B + \delta e_m e_n^T\right)$.

Proof: Exercise.

With this proposition, we can assume without loss of generality that $A, B > 0$ entry-wise. Then, α and β must be positive, so we can scale by α and β to get

$$
\begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 0 & -A \\ -B^T & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} e_m \\ e_n \end{bmatrix}
$$

and $0 \leq [s; t] \perp [u; v] \geq 0$ so we have an LCP!

Theorem 1 *(Relation of NE and LCP) Assume $A, B > 0$. Then if $(\bar{x}, \bar{y})$ is a NE of (A, B), $\left(\bar{u} = \bar{x}/\bar{x}^T B\bar{y}, \bar{v} = \bar{y}/\bar{x}^T A\bar{y}\right)$ with suitable $(\bar{s}, \bar{t})$ is a complementary solution to the LCP above.*

Conversely, if $(\bar{s}, \bar{t}, \bar{u}, \bar{v})$ is a complementary solution to the LCP above, and $(\bar{u}, \bar{v}) \neq (0, 0)$, *then $\left(\bar{x} = \bar{u}/e_m^T\bar{u}, \bar{y} = \bar{v}/e_n^T\bar{v}\right)$ is a NE for (A, B).*

Proof: We have already shown the first part.

For the converse, note that if $\bar{u} \neq 0$, some $\bar{u}_i > 0$ so some $\bar{s}_i = 0$, so $\bar{v} \neq 0$. Similarly, $\bar{v} \neq 0$ implies $\bar{u} \neq 0$. So $\bar{x}$ and $\bar{y}$ are well defined in $X \times Y$. But, $A\bar{v} \leq e_m$, with equality in the ith position if $\bar{u}_i > 0$, implies that $A\bar{y} \leq \alpha e_m$, with $\alpha = 1/e_n^T\bar{v}$, with equality in the ith position if $x_i > 0$.

Similarly, $B^T\bar{x} \leq \beta e_n$ for $\beta = 1/e_m^T\bar{u}$ with equality in the jth position if $\bar{y}_j > 0$. Hence, $(\bar{x}, \bar{y})$ is a NE for (A, B).

Note that $M = \begin{bmatrix} 0 & -A \\ -B^T & 0 \end{bmatrix}$ is neither a P-matrix nor (in general) monotone. It is monotone if $B = -A$, which corresponds to a zero-sum game. Note that the set of *feasible* solutions to this LCP is bounded.

For other applications of LCP, see the paper of Ferris and Pang on the course homepage. Note that the complementary variables in equilibrium problems are the prices of goods and their corresponding excess supplies. Also, see the slides by Mihai Anitescu on the homepage. Here, complementary variables in mechanics problems model non-intersection of bodies (the distance between them is nonnegative) and the corresponding normal force.