Mathematical Programming II	**Lecture 2**
ORIE 6310 Spring 2014	**January 28, 2014**
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Last time, we proved the Karush-Kuhn-Tucker (KKT) theorem.

Theorem 1. *If $\overline{x}$ is a local minimizer for*

$$\begin{aligned} \underset{x\in\mathbf{R}^n}{\text{minimize}} \quad & f(x) \\ \text{subject to} \quad & g(x) \le 0 \end{aligned}$$

and MFCQ holds at $\overline{x}$, then $\exists\ \overline{u} \in \mathbf{R}^m$ such that

$$\begin{aligned} \nabla f(\overline{x}) + \nabla g(\overline{x})\overline{u} = 0 & \\ \overline{u} \ge 0,\ g(\overline{x}) \le 0 & \\ \overline{u}_i g_i(\overline{x}) = 0 & \qquad \forall\ i = 1,\dots,m. \end{aligned}$$

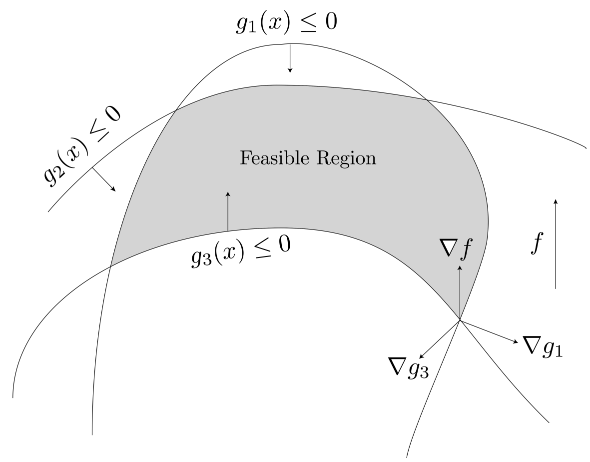

Figure 1: KKT Conditions Geometrically

Remark 1. For linear constrains, MFCQ is not necessary.

Example 1. ***QP (Quadratic Programming)****:* Consider the quadratic program

$$\begin{aligned} \underset{x\in\mathbf{R}^n}{\text{minimize}} \quad & c^\top x + \frac{1}{2}x^\top H x \\ \text{(QP)} \quad \text{subject to} \quad & Ax \ge b \\ & x \ge 0, \end{aligned}$$

where $A \in \mathbf{R}^{m\times n},\ b \in \mathbf{R}^m,\ c \in \mathbf{R}^n$, and $H = H^\top \in \mathbf{R}^{n\times n}$. Note that

$$g(x) = \begin{pmatrix} b - Ax \\ -x \end{pmatrix} \in \mathbf{R}^N, \quad N = m+n,$$

and

$$f(x) = \sum_{j=1}^{n} c_j x_j + \sum_{j=1}^{n} \frac{1}{2} h_{jj} x_j^2 + \sum_{i<j} h_{ij} x_i x_j.$$

Therefore,

$$\begin{aligned} \nabla f(x) &= c + Hx && \in \mathbf{R}^n, \\ \nabla g(x) &= [-A^\top, -I] && \in \mathbf{R}^{n \times N}. \end{aligned}$$

Let $\overline{u} = (\overline{y}; \overline{s})$. The KKT conditions are:

$$\begin{aligned} c + H\overline{x} - A^\top \overline{y} - \overline{s} = 0 & \\ \overline{y} \geq 0, \quad \overline{s} \geq 0, \quad b - A\overline{x} \leq 0, \quad -\overline{x} \leq 0 & \\ \overline{y}_i (b - A\overline{x})_i = 0 & \qquad \forall\ i = 1, \ldots, m \\ \overline{s}_j (-\overline{x}_j) = 0 & \qquad \forall\ j = 1, \ldots, n. \end{aligned}$$

Let $\overline{t} := A\overline{x} - b$. Then, $(\overline{s}, \overline{t}, \overline{x}, \overline{y})$ solves:

$$\begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} H & -A^\top \\ A & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} c \\ -b \end{pmatrix}$$

$$\begin{pmatrix} s \\ t \end{pmatrix} \geq 0, \quad \begin{pmatrix} x \\ y \end{pmatrix} \geq 0, \quad \begin{pmatrix} s \\ t \end{pmatrix}_k \begin{pmatrix} x \\ y \end{pmatrix}_k = 0 \ \ \forall\ k.$$

This is an instance of the **Linear Complementarity Problem (LCP)**:

$$w = Mz + q, \qquad w \geq 0, \qquad z \geq 0, \qquad w \cdot z = 0,$$

with

$$M = \begin{pmatrix} H & -A^\top \\ A & 0 \end{pmatrix} \in \mathbf{R}^{N \times N}, \qquad q = \begin{pmatrix} c \\ -b \end{pmatrix} \in \mathbf{R}^N.$$

Note that we can interpret $w \cdot z$ as the Hadamard (componentwise) product:

$$\begin{pmatrix} w_1 z_1 \\ \vdots \\ w_N z_N \end{pmatrix}$$

of w and z. With $w, z \geq 0$, this is equivalent to $w^\top z = 0$. This is often written as:

$$0 \leq w \perp z \geq 0.$$

The LCP

$$w = Mz + q \tag{1}$$

$$w \geq 0,\ \ z \geq 0 \tag{2}$$

$$w \cdot z = 0 \tag{3}$$

is denoted by LCP(M, q) and is called feasible if there exists w, z satisfying (1) and (2), and then we call (w, z) feasible for the LCP. We call (w, z) complementary if it also satisfies (3). The properties of LCP(M, q) depend heavily on the properties of the matrix M.

Definition 1. *$M \in \mathbf{R}^{N\times N}$ is a **P-matrix** if all its principal minors are positive, i.e.,* $\det(M_{JJ}) > 0 \ \forall \ J \subset \{1, \ldots, N\}$.

Definition 2. *$M \in \mathbf{R}^{N\times N}$ is **monotone** if $z^\top M z \geq 0$ for all $z \in \mathbf{R}^N$, i.e., $\frac{1}{2}(M + M^\top)$ is symmetric positive semidefinite (PSD).*

Remark 2. $M = \begin{pmatrix} H & -A^\top \\ A & 0 \end{pmatrix}$ is not a P-matrix if $m > 0$ but is monotone as long as H is PSD:

$$\begin{pmatrix} x \\ y \end{pmatrix}^\top \begin{pmatrix} H & -A^\top \\ A & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = x^\top H x \geq 0.$$

Also note that:

$$M = \begin{pmatrix} H & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -A^\top \\ A^\top & 0 \end{pmatrix},$$

where the first matrix on the right-hand-side is positive semidefinite and the second is skew symmetric.

Remark 3. Note that LCP asks for us to write q as a non-negative linear combination of a complementary set of columns of $[I, -M]$ (i.e., we cannot use both the jth column of I and the jth column of $-M$ for any j).

Example 2. $(N = 2)$. Let $I = \begin{pmatrix} e_1 & e_2 \end{pmatrix}$ and $M = \begin{pmatrix} m_1 & m_2 \end{pmatrix}$. Note that:

$$w = Mz + q \iff q = Iw - Mz$$

Refer to Figure 2 for the following examples. The complementary cones (sets of nonnegative combinations of complementary sets of columns) are marked.

(a) $M = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$.

   M is a P-matrix, but it's not monotone. Complementary cones form a partition of $\mathbf{R}^2$. Unique complementary solution for all q.

(b) $M = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

   M is monotone and psd but not a P-matrix. The problem is not feasible for all q. It is feasible if and only if $q_2 \geq 0$. But it has a complementary solution for every q for which it is feasible (not necessarily unique).

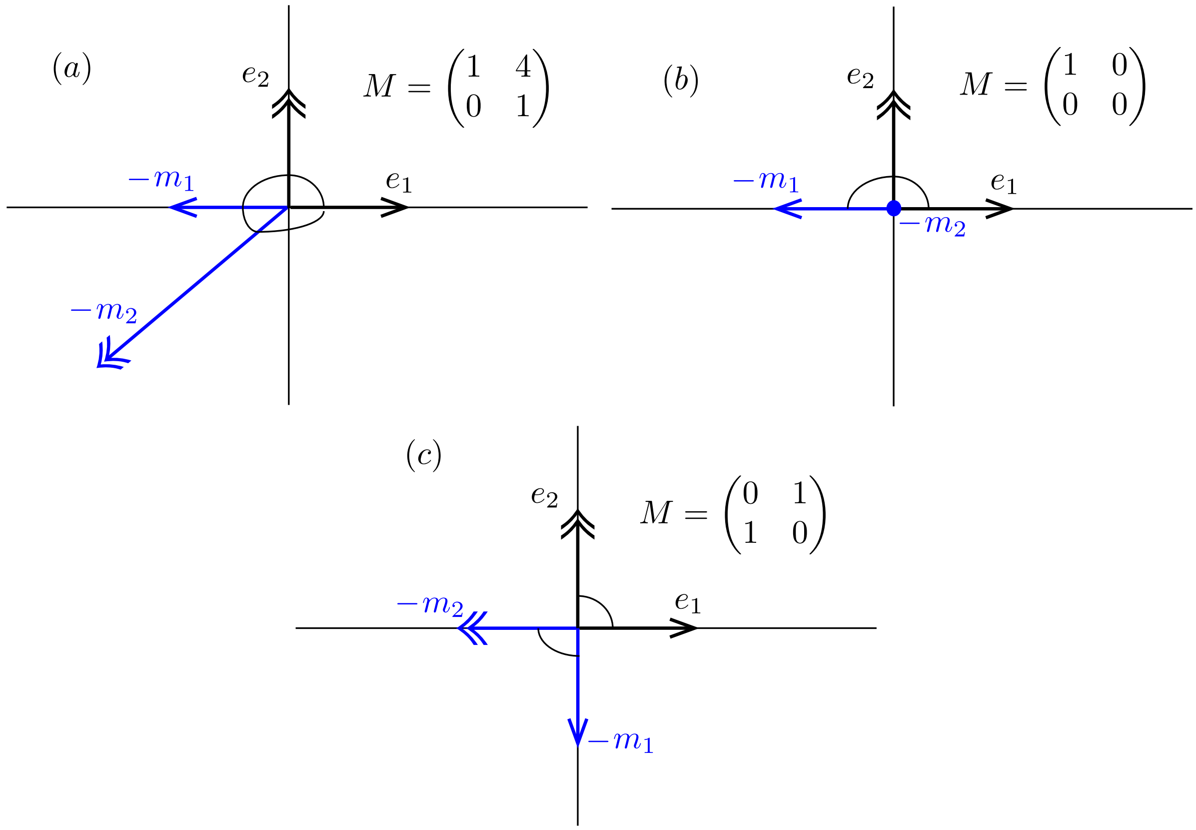

Figure 2: Complementary vectors for different M matrices.

(c) $M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

M is not a P-matrix and it's also not monotone. The problem is feasible for all q and it has a complementary solution only if $q_1 \cdot q_2 \geq 0$.

Significance of matrix classes:

Theorem 2. *If M is a P-matrix, then the LCP(M, q) has a unique complementary solution for every q. (So the complementary cones partition $\mathbf{R}^N$.)*

Proof. We will omit the proof. See Cottle, Pang, and Stone (on reserve in Uris). □

Theorem 3. *If M is monotone, then the LCP(M, q) has a complementary solution whenever it has a feasible solution.*

Proof. We will prove this later using an algorithm. □

Consider again the LCP arising from (QP) with

$$M = \begin{pmatrix} H & -A^\top \\ A & 0 \end{pmatrix}, \quad q = \begin{pmatrix} c \\ -b \end{pmatrix}.$$

In this case, a local minimizer gives a complementary solution to the LCP. Conversely, suppose that $(\overline{s}, \overline{t}, \overline{x}, \overline{y})$ is a complementary solution to the LCP. Then,

$$\begin{aligned} &\overline{s} = H\overline{x} - A^\top \overline{y} + c \\ &\overline{t} = A\overline{x} - b \\ &\overline{t} \cdot \overline{y} = 0, \quad \overline{s} \cdot \overline{x} = 0 \\ &\overline{s},\ \overline{t},\ \overline{x},\ \overline{y} \geq 0. \end{aligned}$$

Therefore, we can write:

$$0 = \overline{s}^\top \overline{x} = \overline{x}^\top H\overline{x} - \overline{x}^\top A^\top \overline{y} + c^\top \overline{x} \tag{4}$$

$$0 = \overline{t}^\top \overline{y} = \overline{y}^\top A\overline{x} - b^\top \overline{y}. \tag{5}$$

Adding equations (4) and (5) we get,

$$\begin{aligned} &\overline{x}^\top H\overline{x} + c^\top \overline{x} - b^\top \overline{y} = 0, \text{ and so} \\ &c^\top \overline{x} + \frac{1}{2}\overline{x}^\top H\overline{x} = b^\top \overline{y} - \frac{1}{2}\overline{x}^\top H\overline{x}. \end{aligned}$$

Theorem 4. *If H is positive semidefinite, then $\overline{x}$ is a global minimizer for (QP).*

Proof. Consider the following dual quadratic programming problem, denoted by (QD):

$$\text{(QD)} \quad \begin{aligned} \underset{y \in \mathrm{R}^m,\ v \in \mathrm{R}^n}{\text{maximize}} \quad & b^\top y - \frac{1}{2}v^\top Hv \\ \text{subject to} \quad & A^\top y - Hv \leq c \\ & y \geq 0. \end{aligned}$$

Note that $(\overline{y}, \overline{x})$ is feasible in (QD) and $\overline{x}$ is feasible in (QP) with the same objective values. Consider any feasible solution x to (QP) (with surplus variable t) and (y, v) to (QD) (with slack variable s). Then,

$$\begin{aligned} c^\top x + \frac{1}{2}x^\top Hx - \left(b^\top y - \frac{1}{2}v^\top Hv\right) &= (A^\top y - Hv + s)^\top x - (Ax - t)^\top y + \frac{1}{2}x^\top Hx + \frac{1}{2}v^\top Hv \\ &= s^\top x + t^\top y + \frac{1}{2}x^\top Hx - x^\top Hv + \frac{1}{2}v^\top Hv \\ &= s^\top x + t^\top y + \frac{1}{2}(x - v)^\top H(x - v) \geq 0. \end{aligned}$$

Hence weak duality holds and so $\overline{x}$ is optimal. $\square$

Digression If we remove the restriction that H is positive semidefinite, we can still show that

$$c^\top \overline{x} - b^T \overline{y} + \overline{x}^T H \overline{x} = 0$$
$$c^\top \overline{x} + \frac{1}{2}\overline{x}^\top H \overline{x} = \frac{1}{2} c^\top \overline{x} + \frac{1}{2} b^\top \overline{y}.$$

Hence, $\overline{x}$ is a local minimizer for (QP) with objective value at most δ, if with some $\overline{y}, \overline{\zeta}, \overline{s}, \overline{t}, \overline{v}$ it is a solution to

$$\begin{pmatrix} s \\ t \\ v \end{pmatrix} = \begin{pmatrix} H & -A^\top & 0 \\ A & 0 & 0 \\ -\frac{1}{2}c^\top & -\frac{1}{2}b^\top & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ \zeta \end{pmatrix} + \begin{pmatrix} c \\ -b \\ \delta \end{pmatrix},$$
$$0 \leq (s; t; v) \perp (x; y; \zeta) \geq 0.$$

So, if we could "solve" arbitrary LCPs, we could find globally optimal solutions to arbitrary QPs, which is hard: consider

$$\begin{aligned} \underset{x \in \mathbf{R}^n}{\text{minimize}} \quad & c^\top x \\ \text{subject to} \quad & Ax \geq b \\ & x \in \{0, 1\}^n, \end{aligned}$$

which is clearly related to

$$\begin{aligned} \underset{x \in \mathbf{R}^n}{\text{minimize}} \quad & c^\top x + \frac{1}{2}\nu x^\top (e - x), \\ \text{subject to} \quad & Ax \geq b \\ & 0 \leq x \leq e, \end{aligned}$$

where $e = (1; \ldots; 1) \in \mathbf{R}^n$ and ν is sufficiently large.

Note: from now on, we shall use n for the dimension of the LCP instead of N, and so A in LP and QP will be $m \times p$, with $n = m + p$. So the primal problem will have p nonnegative variables subject to m general inequality constraints.