Leonid Khachiyan's Contributions to Mathematical Programming
Beyond $LP \in P$

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Outline

- Complexity of convex programming problems
- Matrix scaling
- Block-structured problems and fast approximation algorithms
- Boolean formulae and enumeration
- Geometric problems

(Also note the Khachiyan memorial sessions, TH{1,2,3}R8, with talks by A. Nemirovski, A. Sebo, S. Kale, M. Eleazar, J. Bioch, G. Rudolf, I. Barany, B. Kalantari, and T. Stephen.)

Convex Programming

- QP: M.K. Kozlov, S.P. Tarasov, and Khachiyan gave a polynomial algorithm for convex QP with rational data, using the ellipsoid method.
- Convex Polynomial Programming (ICM, 1983)

Consider the problem

$$\begin{array}{llll} \min & f_0(x) & & \\ (P) & f_i(x) & \leq & 0, \quad i = 1, \ldots, m \\ & x & \in & S \subseteq \mathbb{R}^n, \end{array}$$

where the f_j's are convex polynomials with integer coefficients and maximum degree d, fixed. Let L be the size of the input. S can be $\mathbb{R}^n$, $\mathbb{Z}^n$, or $\mathbb{R}^k \times \mathbb{Z}^{n-k}$.

Results

- If (P) is feasible, it has a feasible solution of norm $O(2^{2^{p(L)}})$.
- If (P) has an optimal solution, it has one similarly bounded.

  Ex.: $x_1 \geq h, \quad x_2 \geq x_1^d, \ldots, \quad x_n \geq x_{n-1}^d$.
- If (P) is feasible, it has a feasible solution x of the form (binary expansion):

$$\{x\} = \{y_r\}\underbrace{0 \ldots 0}_{\delta_r}\{y_{r-1}\}\underbrace{0 \ldots 0}_{\delta_{r-1}}\{y_1\}\underbrace{0 \ldots 0}_{\delta_1}\{x_0\},$$

  all bounded by $2^{p(L)}$.
- Determining the feasibility of (P) with $S = \mathbb{Z}^n$ is NP-complete, and can be done in exponential time.

- For $S = \mathbb{R}^n$, there is a polynomial algorithm for determining an ϵ-feasible solution.
- For fixed (d and) n, there is an exact polynomial algorithm (extends H.W. Lenstra's result).

Extension (Khachiyan-L. Porkolab, 1997):

Let $F(y)$ be a first-order formula over the reals involving polynomials with integer coefficients of degree at most $d \geq 2$. Then:

- If $Y := \{y \in \mathbb{R}^n : F(y) \text{ true}\}$ is convex, for any fixed number of variables (free and quantified), there is a polynomial algorithm that either finds an integral point in Y or determines that no such point exists.

Example: Integer Semidefinite Programming.

General convex programming

Now consider the general problem:

$$\min_{x \in \mathbb{R}^n} \quad f(x), \qquad -e \leq x \leq e,$$

where f is convex and given by an oracle. D.B. Yudin and A.S. Nemirovski (1976) showed that an ϵ-optimal solution can be found in

- $O(n \ln \frac{1}{\epsilon})$ calls to the oracle (method of centers of gravity (MCG)), or
- $O(n^2 \ln \frac{1}{\epsilon})$ calls to the oracle (ellipsoid method).

S.P. Tarasov, I.I. Erlikh, and Khachiyan (1988) developed the implementable Method of Inscribed Ellipsoids (MIE) with the former complexity.

MIE

Each iteration of the MIE proceeds as follows: Given the current localization, a polytope P with $O(n \ln n)$ facets,

a) find the center $\bar{x}$ of an approximate maximum-volume ellipsoid inscribed in P;

b) call the oracle at $\bar{x}$ to get a subgradient $g(\bar{x})$; and

c) Update $P \leftarrow \{x \in P : g(\bar{x})^T(x - \bar{x}) \leq 0\}$.

(MCG is similar, except that in (a), $\bar{x}$ is chosen to be an approximate center of gravity of P.)

In MIE, step (a) can be done using, e.g., the ellipsoid method in polytime.

Matrix Scaling

Given $A \in \mathbb{R}^{n \times n}$, symmetric and positive semidefinite, the problem is to find a scaling $X = \text{Diag}(x)$, $x > 0$, so that

$$XAXe = e,$$

or to show that none exists. In the latter case, find $x \geq 0$, $x \neq 0$, with $x^T Ax = 0$ (this is equivalent to LP).

- Khachiyan and B. Kalantari provide an interior-point method for this problem, which is polynomial if A is rational.
- By contrast, Khachiyan shows that it is NP-hard to determine the scalability of A if the positive definiteness condition is relaxed.

Block-Structured Problems

In a sequence of papers with M. Grigoriadis, Khachiyan studied block-structured problems and interior-point and (price-directive) decomposition methods for their approximate solution. As one general example, consider

$$\min \lambda, \quad \sum_{k=1}^{K} f^k(x^k) \le \lambda e,\ x^k \in B^k, k = 1, \ldots, K,$$

where each $f^k : B^k \to \mathbb{R}^m$ is a nonnegative continuous convex function, and each B^k is a nonempty convex compact set. This is a block-angular resource-sharing problem, but it is strongly related via binary search to the optimization of a separable convex function subject to similar constraints. A special case is the K-commodity minimum-cost network-flow problem.

Approximation methods

Grigoriadis and Khachiyan consider PDD methods that at each coordination step compute a price vector p and then solve solve the K problems of minimizing $p^T f^k(x^k)$ over B^k or over $\{x^k \in B^k : f^k(x^k) \leq \mu e\}$ for some μ; then the solution x and the prices p are updated.

- There is a logarithmic-potential-based unrestricted PDD method that solves (P) to relative accuracy ϵ in $\tilde{O}(\epsilon^{-2} M)$ coordination steps.
- There is an exponential-potential-based restricted PDD method requiring $\tilde{O}(\epsilon^{-2} K \ln M)$ coordination steps.
- These bounds are close to optimal in terms of K and M.
- Hence K-commodity minimum-cost network flow problems on networks with N nodes and M arcs can be approximately solved in $\tilde{O}(\epsilon^{-2} K M^2)$ time (later improved to $\tilde{O}(\epsilon^{-2} K N M)$ time).

Boolean Expressions and Enumeration

Let $f(x)$ and $g(x)$ be two monotone Boolean functions given in disjunctive normal form, e.g.,

$f(x_1, x_2, x_3) = (x_2 \wedge x_3) \vee (x_1 \wedge x_3) \vee (x_1 \wedge x_2)$ and $g(x_1, x_2, x_3) = x_1 \vee (x_2 \wedge x_3)$.

We want to test if f and g are dual, i.e.,

$$g(x) = \bar{f}(\bar{x}) \text{ for all } x \in \{0, 1\}^N,$$

and if not, find an x with $g(x) = f(\bar{x})$. For example, the above f and g are not dual, as witnessed by $x = (1, 0, 0)$, but f is self-dual.

M. Fredman and Khachiyan (1996) showed that this problem can be solved in quasi-polynomial $n^{o(\log n)}$ time, where n is the total number of clauses in f and g.

Enumeration problems

Khachiyan, with E. Boros, K. Elbassioni, V. Gurvich, and K. Makino, studied a number of enumeration problems.

Let $S := \{x \in \mathbb{Z}^n : Ax \geq b,\ 0 \leq x \leq c\}$ be monotone. Given $X \subseteq S$, we want to generate a new member x of S, or show that no such x exists. Khachiyan and his co-authors showed how to do this in time quasi-polynomial in r, n, and $|X|$. This is related to enumerating solutions to the knapsack and multi-knapsack problems, and also to hypergraph dualization.

Let M be a matroid on ground set E, and let $A \subseteq E$. We want to enumerate minimal subsets that span A. Khachiyan and his co-authors show that this can be done in incremental polynomial time. As examples, hyperplanes and circuits of a matroid and generalized Steiner trees and multiway cuts in a graph can also be enumerated in incremental polynomial time.

Geometric Problems

- For any convex body C in $\mathbb{R}^n$, let $w(C)$ denote the volume of a maximum-volume ellipsoid inscribed in C. If a hyperplane through the center of this ellipsoid splits C into C^+ and C^-, then $w(C^+)/w(C) \leq .844$. (By contrast, if we replace maximum-volume inscribed ellipsoid by minimum-volume circumscribing ellipsoid, the bound becomes $1 - O(1/n)$. If we use volume and put the hyperplane through the center of gravity, the bound improves to $1 - (n/(n+1))^n < 1 - 1/e \approx .632$.)
- A. Karzanov and Khachiyan showed that, if C is divided into C^+ and C^- by a surface S, then

$$\text{area}(S) \geq \min[\text{vol}(C^+), \text{vol}(C^-)]/\text{diam}(C).$$

They applied this to estimating the mixing time of certain Markov chains.

- Khachiyan and M. Todd showed that a maximum-volume ellipsoid inscribed in an n-polytope with m facets can be approximated in $\tilde{O}(m^{3.5})$ arithmetical operations using an interior-point method.
- Khachiyan analyzed an algorithm of Fedorov to show that an ϵ-rounding of an n-polytope with m vertices can be computed in $\tilde{O}(\epsilon^{-1}mn^2)$ arithmetical operations and hence an ϵ-optimal minimum-volume circumscribing ellipsoid in $\tilde{O}(\epsilon^{-1}mn^3)$ arithmetical operations. (Related papers appeared in TU1R14.)