ORIE 6310: Mathematical Programming II. Spring 2014.

Take-home Final Exam. Available 9 am Monday, May 12th, due noon Tuesday, May 13th, from Tara Bennett or Sheri Minarski in Rhodes 206.

Items in parentheses — () — below are either explanatory or informational; no work is required from you on these. This is to be *all your own work*. You may use any result from class or homeworks, and any standard real analysis or linear algebra result. Do not consult any other reference, living, dead, or digital. I can dispense hints to help you make progress if you are stuck. (Office: Rhodes 229, 255-9135; mjt7@cornell.edu; home 257-3344, cell 592-1931, before 10.) Good luck!

1. (20 points: 4,5,5,6.)

a) Show that, if M is symmetric positive semidefinite, it is also copositive plus (i.e., $v^T M v \geq 0$ for any $v \geq 0$, with equality only if $(M + M^T)v = 0$).

b) Suppose that M is symmetric positive semidefinite. Prove that, if (w, z) and (w', z') are both complementary solutions of the LCP (M, q), then $w = w'$. Show that this is not true if M is not symmetric, but still monotone, by considering, e.g., $M = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$, $q = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

c) Now suppose M is strictly monotone. Show that, for any q, there is at most one complementary solution to the LCP (M, q).

d) Suppose Lemke's algorithm (complementary pivot algorithm 2) is applied to an LCP (M, q) with M strictly monotone. Prove that the artificial variable z_0 is strictly decreasing throughout the algorithm, except in degenerate pivots.

2. (25 points: 4,8,7,6.)

Consider a polytope $P \subseteq \Re^d$ with n facets, and suppose

$$P = \{y \in \Re^d : A^T y \leq c,\ a^T y \leq \gamma\},$$

with $A \in \Re^{d \times (n-1)}$, $c \in \Re^{n-1}$, $a \in \Re^d$, and $\gamma \in \Re$. Consider the polyhedron in $\Re^{d+1}$ defined by

$$\bar{P} := \{(y; \eta) \in \Re^{d+1} : A^T y \leq c,\ a^T y + \eta \leq \gamma,\ -\eta \leq 0\},$$

with at most (in fact, exactly) $n + 1$ facets. (This is called the *wedge* of P over the facet $\{y \in P : a^T y = \gamma\}$.)

a) Show that the projection $\{y \in \Re^d : (y; \eta) \in \bar{P}$ for some $\eta \in \Re\}$ of $\bar{P}$ is exactly P, and hence that $\bar{P}$ is a polytope.

b) Show that, if y is a vertex of P, then $(y; 0)$ is a vertex of $\bar{P}$, and if $a^T y < \gamma$, so is $(y; \gamma - a^T y)$. Show that all vertices of $\bar{P}$ arise in this way.

c) Suppose y and y' are vertices of P, with $[y, y']$ an edge of P. Show that $[(y; 0), (y'; 0)]$ and $[(y; \gamma - a^T y), (y'; \gamma - a^T y')]$ are edges of $\bar{P}$ (maybe the same one!), and if $a^T y < \gamma$, so is $[(y; 0), (y; \gamma - a^T \hat{y})]$. (In fact, these are the only edges of $\bar{P}$; you can use this, but you don't have to prove it.)

d) Prove that $\delta(d + 1, n + 1) \geq \delta(d, n)$, where $\delta(d, n)$ is the maximum diameter of a polytope in $\Re^d$ with n facets.

3. (30 points: 8,6,8,8.)

a) Suppose that $E(B_+, y_+) \subset \Re^n$ is the minimum volume ellipsoid containing

$$\{x \in E(B,y) : a^T x \le a^T y - \alpha(a^T Ba)^{\frac{1}{2}}\},$$

where $\alpha > -1/n$ and $0 \ne a \in \Re^n$. Show that

$$a^T y - \alpha(a^T Ba)^{\frac{1}{2}} = a^T y_+ + \frac{1}{n}(a^T B_+ a)^{\frac{1}{2}},$$

i.e., the "depth" α of the constraint that was used to make the cut is exactly $-1/n$ in the new ellipsoid.

b) Suppose we apply the ellipsoid method to try to find a point in

$$\{x \in \Re^2 : -e_1^T x \le -1,\, e_1^T x \le 1,\, -e_2^T x \le -\frac{1}{2},\, e_2^T x \le \frac{3}{2}\},$$

starting with $E_0 := \{x \in \Re^2 : \|x\|_2 \le 2\}$. Here e_j denotes the jth unit vector in $\Re^2$. At each iteration, we choose as the cut to define the new ellipsoid the constraint $a_i^T x \le b_i$ with maximum depth

$$\alpha_i := \frac{a_i^T x_k - b_i}{(a_i^T B a_i)^{\frac{1}{2}}},$$

stopping if all α_i's are nonpositive, and using the deep cut method (i.e., the ellipsoid is updated as in (a)).

(i) What are the depths of all the constraints, and what cut is chosen, at the first iteration?

(ii) What are the depths of all the constraints, and what cut is chosen, at the second iteration?

(iii) Show that the algorithm never terminates. (This is not too surprising, as the volume of the feasible region is zero.)

(iv) Assuming that the centers x_k converge to some x_* (you don't need to prove this), show that x_* is not feasible.

4. (25 points: 8,10,7.)

Suppose $w_i > 0$ and $a_i \in \Re^m$ for $1 \le i \le n$. Consider the problem (P):

$$\min \quad \sum_{i=1}^{n} w_i \|y - a_i\|_{p_i}$$

over all $y \in \Re^m$, where each p_i is between 1 and ∞. (If all the p_i's are 2, this is the Fermat-Weber problem.)

a) Rewrite this as a conic programming problem in dual form.

b) Find the dual of the problem in (a). State a strong duality result for these problems and explain why it holds.

c) Now assume $n = 3$, $m = 2$, $w_1 = w_2 = w_3 = p_1 = p_2 = p_3 = 1$, and $a_i = (1; 2), a_2 = (2; 1)$, and $a_3 = (4; 4)$. Find an optimal solution to the original problem (P) and prove it optimal. (In fact, it is easy to find a closed-form solution for any n and m if all p_i's are 1.)