Sensitivity Analysis in LP and SDP Using Interior-Point Methods

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Part I: Linear Programming (LP)

LP in standard form:

$$
\begin{array}{lllrcl}
LPP(b,c) & \min_x & c^T x & & & \\
& \text{s.t.} & & & & \\
& & & Ax & = & b, \\
& & & x & \geq & 0,
\end{array}
$$

c and $x \in IR^n$, $b \in IR^m$, and $A \in IR^{m \times n}$.

Associated dual LP:

$$
\begin{array}{llllllll}
LPD(b,c) & \max_{y,s} & b^T y & & & & & \\
& \text{s.t.} & & & & & & \\
& & & A^T y & + & s & = & c, \\
& & & & & s & \geq & 0,
\end{array}
$$

$y \in IR^m$ and $s \in IR^n$.

- The coefficient matrix A is fixed.

Assumptions:

1. Both $LPP(b,c)$ and $LPD(b,c)$ have strictly feasible points.

2. A has full row rank.

Central Path

Solution to:

$$\begin{array}{ccccccc} & Ax & & & & = & b, \\ A^T y & & & + & s & = & c, \\ & & & XSe & & = & \mu e, \end{array}$$

with $x > 0$ and $s > 0$ for some $\mu > 0$.

Under Assumptions 1 and 2, such a solution exists and is unique for each positive μ.

- Let (x, y, s) be the current iterate.

- Seek an approximation to the point on the central path with parameter μ.

Using Newton's method:

$$\begin{array}{rcrcl} & A\Delta x & & & = & r_p \\ A^T\Delta y & & + & \Delta s & = & r_d \\ & S\Delta x & + & X\Delta s & = & r_{xs}, \end{array} \tag{1}$$

where

- $r_p = b - Ax$,

- $r_d = c - A^T y - s$, and

- $r_{xs} = \mu e - XSe$.

Definition. *A Newton step from (x, y, s)* **targeting** *the feasible pair of points (x', y', s') of $LPP(b', c')$ and $LPD(b', c')$ that satisfies $X'S'e = v$ is the triple $(\Delta x, \Delta y, \Delta s)$ solving (1) for $r_p = b' - Ax$, $r_d = c' - A^T y - s$, and $r_{xs} = v - XSe$.*

Remark 1. *Such a pair of points (x', y', s') might not exist, but the Newton step is still well-defined.*

Proposition 1. *Assume:*

- (x, y, s) *is a strictly feasible pair of points for* $LPP(b, c)$ *and* $LPD(b, c)$*,*

- b *is replaced by* $b' = b + \Delta b$*, where* $\Delta b \in IR^m$*, and*

- *a full Newton step is taken from* (x, y, s) *targeting the feasible pair of points* (x', y', s') *of* $LPP(b', c)$ *and* $LPD(b', c)$ *that satisfies* $X'S'e = XSe$*.*

If, and only if,

$$\|S^{-1}A^T(AS^{-1}XA^T)^{-1}\Delta b\|_\infty \leq 1,$$

- *then resulting iterate will be feasible for the new problem.*

Moreover, duality gap will decrease.

Proof:

$r_p = \Delta b$, $r_d = 0$, and $r_{xs} = 0$.

The third equation defining the Newton step:

$$S\Delta x + X\Delta s = 0.$$

Rewriting it componentwise,

$$s_i \Delta x_i + x_i \Delta s_i = 0 \quad \text{so} \quad \frac{\Delta x_i}{x_i} + \frac{\Delta s_i}{s_i} = 0, \qquad i = 1, \ldots, n.$$

Proof (Cont'd):

The next iterate will be feasible iff

$$x_i + \Delta x_i \geq 0, \quad \text{and} \quad s_i + \Delta s_i \geq 0, \quad i = 1, \ldots, n.$$

Combining two observations, it is necessary and sufficient to have:

$$\left\| S^{-1} \Delta s \right\|_{\infty} \leq 1.$$

The result follows from the solution of the Newton step.

□

Proposition 2. *Assume:*

- (x, y, s) *is a strictly feasible pair of points for* $LPP(b, c)$ *and* $LPD(b, c)$*,*
- c *is replaced by* $c' = c + \Delta c$*, where* $\Delta c \in IR^n$*, and*
- *a full Newton step is taken from* (x, y, s) *targeting the feasible pair* (x', y', s') *of* $LPP(b, c')$ *and* $LPD(b, c')$ *that satisfies* $X'S'e = XSe$*.*

If, and only if,

$$\|S^{-1}(I - A^T(AS^{-1}XA^T)^{-1}AS^{-1}X)\Delta c\|_\infty \leq 1,$$

- *then resulting iterate will be feasible for the new problem.*

Moreover, duality gap will decrease.

Simplex Approach

- Partition A as B and N corresponding to the basic and nonbasic columns, respectively.

- Let (x^*, y^*, s^*) be a pair of optimal solutions for $LPP(b, c)$ and $LPD(b, c)$.

- Partition x^* as x_B^* and x_N^*; s^* as s_B^* and s_N^*; c as c_B and c_N accordingly.

1. Let b be replaced by $b + \Delta b$. The optimal basis remains optimal iff

$$B^{-1}(b + \Delta b) \geq 0 \quad \text{or} \quad B^{-1}\Delta b \geq -B^{-1}b = -x_B^*.$$

2. Let c be replaced by $c + \Delta c$. The optimal basis remains optimal iff

$$c_N^T + \Delta c_N^T - c_B^T B^{-1} N - \Delta c_B^T B^{-1} N \geq 0 \quad \text{or} \quad \Delta c_N - N^T B^{-T} \Delta c_B \geq -s_N^*.$$

Assumption: The optimal solution is unique and non-degenerate.

Symmetrized simplex bounds

1. Change in b

$$B^{-1}\Delta b \geq -B^{-1}b = -x_B^* \quad \text{so} \quad (X_B^*)^{-1}B^{-1}\Delta b \geq -e.$$

The largest L_∞-box around the origin:

$$\|(X_B^*)^{-1}B^{-1}\Delta b\|_\infty \leq 1.$$

Symmetrized simplex bounds (Cont'd):

2. Change in c

$$\Delta c_N - N^T B^{-T} \Delta c_B \geq -s_N^* \quad \text{so} \quad (S_N^*)^{-1}(\Delta c_N - N^T B^{-T} \Delta c_B) \geq -e.$$

The largest L_∞-box around the origin:

$$\|(S_N^*)^{-1}(\Delta c_N - N^T B^{-T} \Delta c_B)\|_\infty \leq 1.$$

Comparison

Theorem 1. *Let (x, y, s) be any pair of strictly feasible solutions with small duality gap μn. The bounds arising from Propositions 1 and 2 evaluated at these points are given by*

$$\left\| \begin{bmatrix} (X_B^*)^{-1}B^{-1} + O(\mu) \\ O(\mu) \end{bmatrix} \Delta b \right\|_\infty \le 1,$$

and

$$\left\| \begin{bmatrix} O(\mu) & O(\mu) \\ -(S_N^*)^{-1}N^T B^{-T} + O(\mu) & (S_N^*)^{-1} + O(\mu) \end{bmatrix} \begin{bmatrix} \Delta c_B \\ \Delta c_N \end{bmatrix} \right\|_\infty \le 1.$$

Moreover, these bounds are asymptotically the same as the symmetrized simplex bounds.

Conclusion of Part I

Under the assumption of unique, non-degenerate solution, the interior-point approach yields asymptotically exactly the same bounds as those arising from the simplex approach that keep the current basis optimal (after symmetrization with respect to the origin).

Part II: Semidefinite Programming (SDP)

SDP in standard form:

$$\begin{array}{lll} SDP(b,C) & \min_X & C \bullet X \\ & & \mathcal{A}X = b, \\ & & X \succeq 0, \end{array}$$

where

$$\mathcal{A}X = (A_i \bullet X)_{i=1}^m,$$

$A_i \in SIR^{n\times n}$, $b \in IR^m$, $C \in SIR^{n\times n}$ are given, and $X \in SIR^{n\times n}$.

$SIR^{n\times n}$: $n \times n$ symmetric matrices.

$X \succeq 0$: X is symmetric positive semidefinite.

$P \bullet Q = \mathrm{Trace}\,(P^T Q) = \sum_{ij} P_{ij} Q_{ij}$

Associated dual problem:

$$\begin{array}{llrcrcl} SDD(b,C) & \quad \max_{y,S} & b^T y & & & & \\ & & \mathcal{A}^* y & + & S & = & C, \\ & & & & S & \succeq & 0, \end{array}$$

where $y \in IR^m$, $S \in SIR^{n \times n}$, and

$$\mathcal{A}^* y = \sum_{i=1}^{m} y_i A_i.$$

Assumptions:

1. Both $SDP(b,C)$ and $SDD(b,C)$ have strictly feasible solutions.

2. $\mathcal{A}$ is surjective, i.e. A_is are linearly independent.

Central Path

Solution to:

$$
\begin{array}{rcrcl}
 & A_i \bullet X & & = & b_i, \quad \text{for } i = 1, \ldots, m, \\
\sum_{i=1}^m y_i A_i & & + \quad S & = & C, \\
 & XS & & = & \mu I,
\end{array}
\tag{2}
$$

together with $X \succ 0$ and $S \succ 0$ for $\mu > 0$.

Remark. *Newton's method cannot be applied directly to (2) since the system is not square.*

Remedy: Symmetrize the third equation so that the residual lies in $SIR^{n \times n}$.

- Let (X, y, S) be the current iterate.

- The directions we will examine will be Newton steps for nonlinear systems of the form

$$\begin{array}{rcccl} \mathcal{A}\tilde{X} & & & = & b, \\ & \mathcal{A}^*\tilde{y} & + \quad \tilde{S} & = & C, \\ \Theta(\tilde{X}, \tilde{S}) & & & = & \Theta(X', S'), \end{array}$$

where $\Theta(X, S)$ is some symmetrization of XS and where X' and S' are the targeted points.

- More than 20 different symmetrizations!

The Newton step $(\Delta X, \Delta y, \Delta S)$ is the solution to

$$\begin{array}{rcrcrcl} & & \mathcal{A}\Delta X & & & = & r_p, \\ \mathcal{A}^*\Delta y & & & + & \Delta S & = & R_d, \\ & & \mathcal{E}\Delta X & + & \mathcal{F}\Delta S & = & R_{EF}, \end{array}$$

where

- $r_p = b - \mathcal{A}X$ is the primal residual,
- $R_d = C - \mathcal{A}^*y - S$ is the dual residual,
- the operators $\mathcal{E} = \mathcal{E}(X, S)$ and $\mathcal{F} = \mathcal{F}(X, S)$ are the derivatives of Θ with respect to $\tilde{X}$ and $\tilde{S}$ respectively, evaluated at (X, S), and
- $R_{EF} = R_{EF}(X, S) = \Theta(X', S') - \Theta(X, S)$.

The solution to the Newton step is unique if

- $\mathcal{E}$ is nonsingular, and
- $m \times m$ Schur complement matrix $\mathcal{A}\mathcal{E}^{-1}\mathcal{F}\mathcal{A}^*$ is nonsingular.

In this case, Newton step is given by:

$$\begin{aligned} (\mathcal{A}\mathcal{E}^{-1}\mathcal{F}\mathcal{A}^*)\Delta y &= r_p - \mathcal{A}\mathcal{E}^{-1}(R_{EF} - \mathcal{F}R_d), \\ \Delta S &= R_d - \mathcal{A}^*\Delta y, \\ \Delta X &= \mathcal{E}^{-1}(R_{EF} - \mathcal{F}\Delta S). \end{aligned}$$

We consider the following search directions:

- AHO (Alizadeh, Haeberly, Overton)
- H..K..M (Helmberg et. al., Kojima et. al., Monteiro)
- NT (Nesterov, Todd)

Notation:

$$(P \odot Q)K := \frac{1}{2}(PKQ^T + QKP^T),$$

where $P, Q \in IR^{n\times n}$ and $K \in SIR^{n\times n}$, and we will regard it as an operator from $SIR^{n\times n}$ to itself.

Observations:

1. For the three directions,

$$\mathcal{E} = S \odot M, \qquad \mathcal{F} = MX \odot I,$$

   where M is a symmetric positive definite matrix.

   $\mathcal{E}$ is invertible.

   - $M = I$ for AHO
   - $M = S$ for H..K..M
   - $M = W^{-1}$ for NT where $WSW = X$.

2. For all (X, y, S) strictly feasible, $\mathcal{E}^{-1}\mathcal{F}$ is positive definite for H..K..M and NT. This also holds for AHO under additional assumptions on (X, y, S).

Proposition 3. *Assume*

- (X, y, S) *is a strictly feasible pair of points for* $SDP(b, C)$ *and* $SDD(b, C)$*,*
- $\mathcal{A}\mathcal{E}^{-1}\mathcal{F}\mathcal{A}^*$ *is nonsingular,*
- b *is replaced by* $b' = b + \Delta b$*,*
- *a full Newton step is taken from* (X, y, S) *targeting the feasible pair* (X', y', S') *of* $SDP(b', C)$ *and* $SDD(b', C)$ *that satisfies* $\Theta(X', S') = \Theta(X, S)$*.*

If

$$\left\| X^{-\frac{1}{2}} \left(\mathcal{E}^{-1}\mathcal{F}\mathcal{A}^* \left[(\mathcal{A}\mathcal{E}^{-1}\mathcal{F}\mathcal{A}^*)^{-1} \Delta b \right] \right) X^{-\frac{1}{2}} \right\|_2 \leq 1,$$

$$\left\| S^{-\frac{1}{2}} \left(\mathcal{A}^* \left[(\mathcal{A}\mathcal{E}^{-1}\mathcal{F}\mathcal{A}^*)^{-1} \Delta b \right] \right) S^{-\frac{1}{2}} \right\|_2 \leq 1,$$

then the resulting iterate will be feasible for the new problem. Moreover, the duality gap of the new iterate will decrease if $\mathcal{E}^{-1}\mathcal{F}$ *is positive definite.*

Proof:

The next iterate will be feasible for the new problem if and only if $X + \Delta X \succeq 0$ and $S + \Delta S \succeq 0$. But,

$$X + \Delta X \succeq 0 \quad \text{holds iff} \quad I + X^{-\frac{1}{2}} \Delta X X^{-\frac{1}{2}} \succeq 0.$$

Therefore, all eigenvalues of $X^{-\frac{1}{2}} \Delta X X^{-\frac{1}{2}}$ should be ≥ 1. A sufficient condition is

$$\|X^{-\frac{1}{2}} \Delta X X^{-\frac{1}{2}}\|_2 \leq 1.$$

Proof (Cont'd):

Similarly,

$$S + \Delta S \succeq 0 \quad \text{holds iff} \quad I + S^{-\frac{1}{2}} \Delta S S^{-\frac{1}{2}} \succeq 0.$$

A sufficient condition is

$$\|S^{-\frac{1}{2}} \Delta S S^{-\frac{1}{2}}\|_2 \leq 1.$$

The result follows from the solution to the Newton system.

Proposition 4. *Assume:*

- (X, y, S) *is a strictly feasible pair of points for* $SDP(b, C)$ *and* $SDD(b, C)$,

- $\mathcal{A}\mathcal{E}^{-1}\mathcal{F}\mathcal{A}^*$ *is nonsingular,*

- C *is replaced by* $C' = C + \Delta C$, *where* $\Delta C \in SIR^{n \times n}$, *and*

- *a full Newton step is taken from* (X, y, S) *targeting the feasible pair* (X', y', S') *of* $SDP(b, C')$ *and* $SDD(b, C')$ *that satisfies* $\Theta(X', S') = \Theta(X, S)$.

If

$$\left\| X^{-\frac{1}{2}} \left(-\mathcal{E}^{-1}\mathcal{F}\Delta C + \mathcal{E}^{-1}\mathcal{F}\mathcal{A}^*(\mathcal{A}\mathcal{E}^{-1}\mathcal{F}\mathcal{A}^*)^{-1}\mathcal{A}\mathcal{E}^{-1}\mathcal{F}\Delta C \right) X^{-\frac{1}{2}} \right\|_2 \leq 1,$$

$$\left\| S^{-\frac{1}{2}} \left(\Delta C - \mathcal{A}^*(\mathcal{A}\mathcal{E}^{-1}\mathcal{F}\mathcal{A}^*)^{-1}\mathcal{A}\mathcal{E}^{-1}\mathcal{F}\Delta C \right) S^{-\frac{1}{2}} \right\|_2 \leq 1.$$

then the resulting iterate will be feasible for the new problem. Moreover, the duality gap of the new iterate will decrease if $\mathcal{E}^{-1}\mathcal{F}$ *is positive definite.*

Concluding Remarks

- For LP, the bounds arising from the interior-point approach compares nicely with those from the simplex approach under the assumption of unique, non-degenerate solution.
- The comparison in the degenerate case remains unanswered.
- For SDP, our preliminary computational tests indicate that the three directions perform similarly if the iterates lie in a relatively wide neighborhood of the central path.
- NT direction yields similar results even when the iterates get too close to the boundary whereas the other two directions deteriorate significantly.