OR 631: Mathematical Programming II. Spring 2014.
Homework Set 4. Due: Tuesday April 29.

1. Suppose we wish to solve the problem

$$\min f(x), \qquad h_i(x) \le 0,\ i = 1, \ldots, m, \quad x \in B(0,R),$$

and suppose that f is convex and has range at most 1 on $B(0,R)$, that each h_i is convex and is at most 1 on $B(0,R)$, and that the problem has an optimal solution x_*.

Consider the following algorithm, given $\epsilon > 0$. Start with $x_0 = 0$. At iteration k, if $\|x_k\| > R$, choose $v_k = x_k$. If for some i, $h_i(x_k) > \epsilon$, choose any such i and set v_k to be a subgradient of h_i at x_k. If $h_i(x_k) \le \epsilon$ for all i, set v_k to be a subgradient of f at x_k. If $v_k = 0$, stop; otherwise set $x_{k+1} = x_k - \frac{\epsilon R}{\|v_k\|} v_k$.

Show that within ϵ^{-2} iterations, the method will give an x_k with $h_i(x_k) \le \epsilon$ for all i and $f(x_k) \le f(x_*) + \epsilon$.

2. Assume that f is a twice continuously differentiable convex function on $\Re^n$ and that for every x, the eigenvalues of $\nabla^2 f(x)$ are bounded between $\ell > 0$ and $L < \infty$. Show that, for every $x, y \in \Re^n$,

$$f(x) + \nabla f(x)^T (y - x) + \frac{\ell}{2}\|y - x\|^2 \le f(y) \le f(x) + \nabla f(x)^T (y - x) + \frac{L}{2}\|y - x\|^2.$$

3. a) Suppose that g is a convex function on $\Re^n$. Show that, for every $z \in \Re^n$ and $L > 0$, the problem

$$(P(z)) \qquad \min g(x) + \frac{L}{2}\|x - z\|^2$$

has an optimal solution.

b) Assume that, for any $z \in \Re^n$, you can solve $(P(z))$ efficiently. Show how you can solve the problem

$$\min f(x_k) + \nabla f(x_k)^T (x - x_k) + \frac{L}{2}\|x - x_k\|^2 + g(x)$$

efficiently. What is the solution if $g \equiv 0$?

c) Find the solution to $(P(z))$ explicitly if $g(x) := \|x\|_1$.