OR 631: Mathematical Programming II. Spring 2014.
Homework 3 solutions.

1. This question and the next are concerned with central cuts. Suppose we have an ellipsoid $E := E(B, y)$, and we add two cuts symmetrically placed with respect to the center y. Consider

$$\bar{E}_\alpha := \{x \in E : a^T y - \alpha\sqrt{a^T B a} \leq a^T x \leq a^T y + \alpha\sqrt{a^T B a}\}$$

for some nonzero $a \in \Re^n$ and some $0 \leq \alpha \leq 1$.

a) Write the condition for x to lie in $\bar{E}_\alpha$ as two quadratics.

b) By combining these two quadratics suitably, find an ellipsoid $E(B_+, y_+)$ that contains $\bar{E}_\alpha$, depending on a scalar parameter σ.

c) Find the value of σ that minimizes the volume of the resulting ellipsoid as a function of α. Show that for $\alpha = n^{-1/2}$ this ellipsoid is just E, while for α smaller than this it gives an ellipsoid of smaller volume than E. (In fact, this is the minimum-volume ellipsoid among all those containing $\bar{E}_\alpha$, not just those obtained this way.)

a) For simplicity, define $\bar{a}$ as $a/(a^T B a)^{1/2}$, so that the two-sided constraint can be written as $-\alpha \leq a^T(x - y) \leq \alpha$. Then the two quadratic inequalities are

$$\begin{aligned} (x-y)^T B^{-1}(x-y) &\leq 1, \\ (x-y)^T \bar{a}\bar{a}^T(x-y) &\leq \alpha^2. \end{aligned}$$

b) Multiply the first inequality by $1 - \sigma$ and the second by σ for $0 \leq \sigma \leq 1$ and add, to get

$$(x-y)^T((1-\sigma)B^{-1} + \sigma\bar{a}\bar{a}^T)(x-y) \leq 1 - \sigma + \sigma\alpha^2,$$

or, using the Sherman-Morrison-Woodbury formula, $(x-y)^T B_+^{-1}(x-y) \leq 1$, where

$$B_+ := \left(\frac{1-\sigma+\sigma\alpha^2}{1-\sigma}\right)\left(B - \sigma\bar{a}\bar{a}^T\right).$$

(This assumes $\sigma < 1$, since otherwise, the quadratic inequality defines a slab of infinite volume for $n > 1$. For $n = 1$, the problem is trivial and the optimal σ is 1. We assume $n > 1$ in what follows.) With $\sigma < 1$, B_+ is positive definite, and this defines the ellipsoid $E(B_+, y)$ (note that y is unchanged). Since its defining inequality is derived from those for $\bar{E}_\alpha$, $E(B_+, y)$ contains it.

c) The volume of this ellipsoid is the square root of the determinant of B_+ times that of the unit ball, so we want to minimize the determinant of B_+, which is easily seen to be $f(\sigma)$ times that of B, with

$$f(\sigma) := \frac{(1-\sigma+\sigma\alpha^2)^n}{(1-\sigma)^{n-1}}.$$

It is easy to check that the derivative of f, for all $0 \leq \sigma < 1$, is a positive multiple of

$$g(\sigma) := -n(1-\alpha^2)(1-\sigma) + (n-1)(1-\sigma+\sigma\alpha^2).$$

Note that g has a unique root at $\sigma^* = (1 - n\alpha^2)/(1-\alpha)$, and is negative to the left and positive to the right. Hence for $0 \leq \alpha \leq n^{-1/2}$, f is minimized at σ^*, while if $\alpha > n^{-1/2}$, $\sigma^* < 0$ and f is increasing for σ between 0 and 1, and so is minimized by $\sigma = 0$ (and so the minimum-volume ellipsoid of this form is the original ellipsoid E). For $0 \leq \alpha < n^{-1/2}$, f is decreasing from 0 to σ^*, so the resulting ellipsoid has volume strictly smaller than that of E.

2. Consider a centrally symmetric polytope, a bounded polyhedron of the form $P := \{x \in \Re^n : -b \leq Ax \leq b\}$ for some A, b.

a) Show that there is a minimum-volume ellipsoid $E = E(B, y)$ containing P.

b) Show that any such must have $y = 0$, i.e., it must be centrally symmetric also.

c) Show that, if $E(B, 0)$ is a (the) minimum-volume ellipsoid containing P, then $\{n^{-1/2}x : x \in E(B, 0)\}$ is contained in P.

(Hence such polytopes can be rounded to a factor $\sqrt{n}$, not n as in the general case. In fact, this holds for any centrally symmetric convex body.)

a) If P is a convex body, i.e., has a nonempty interior, this follows from Proposition 1 of Lecture 17. If not, P lies in a lower-dimensional affine set. In fact, the affine hull of P is a lower-dimensional affine set, and P has a nonempty interior relative to this; then P can be enclosed in a minimum-volume ellipsoid in this lower-dimensional set, and hence in a degenerate ellipsoid in $\Re^n$ with n-dimensional volume 0, hence clearly minimum! Henceforth assume P is a convex body; otherwise, the following arguments can all be applied within its lower-dimensional affine hull.

b) Suppose $E(B, y)$ is a minimum-volume ellipsoid containing P, with $y \neq 0$. Then $E(B, -y)$ contains $-P = P$. So for any $x \in P$, $(x-y)^T B^{-1}(x-y) \leq 1$ and $(x+y)^T B^{-1}(x+y) \leq 1$. Averaging, we find $x^T B^{-1} x \leq 1 - y^T B^{-1} y$. Thus P is contained in $E(\lambda B, 0)$ with $\lambda := 1 - y^T B^{-1} y < 1$, an ellipsoid of strictly smaller volume. This is a contradiction.

c) Suppose not, so that there is an x in $E(B, 0)$ with $n^{-1/2}x \notin P$. Thus it can be separated from P, so there is an inequality $a^T z \leq \beta$ valid for P with $a^T x > n^{1/2}\beta$. Since $a^T x \leq \sqrt{a^T B a}$, $\beta = \alpha\sqrt{a^T B a}$ for some $\alpha < n^{-1/2}$. But if $a^T z \leq \alpha\sqrt{a^T B a}$ for all $z \in P$, then $a^T z \geq -\alpha\sqrt{a^T B a}$ for all $z \in P$ also, since P is centrally symmetric. So $P \subseteq \bar{E}_\alpha$, implying that P can be enclosed in an ellipsoid of smaller volume by Q1, a contradiction.

3. Suppose that $P := \{x \in \Re^n : A^T x \leq e\}$ is bounded (where e is the vector of ones as usual). Assume that the function $B \to -\ln\det B$ is convex as a function of the entries of the symmetric matrix B.

a) Show how the problem of finding the maximum volume ellipsoid with center y contained in P can be written as an optimization problem with a finite number of constraints. (Argue that the positive semidefiniteness constraint can be eliminated.)

b) Exhibit a feasible solution to this problem.

c) Show that if the center y is restricted to be 0, your optimization problem can be converted to one with linear constraints on B.

d) Now return to the general case, where y is a variable. Try to rewrite the optimization problem with convex constraints (you may want to consider the symmetric square root).

a) We use the ellipsoid $E(B, y)$, whose volume is proportional to $\sqrt{\det B}$. We want every point in the ellipsoid to satisfy each constraint $a_j^T x \leq 1$, where a_j is the jth column of A. This holds as long as $a_j^T y + \sqrt{a_j^T B a_j} \leq 1$. So we can write the problem as

$$\begin{aligned} \min_{B,y} \quad & -\ln\det B \\ a_j^T y + \sqrt{a_j^T B a_j} \;\; & \leq \;\; 1, \qquad \text{for all } j, \\ B \text{ positive definite.} & \end{aligned}$$

We can eliminate the last constraint as follows: Define the function $\text{lndet}(M)$ on symmetric $n \times n$ matrices as $\ln\det(M)$ if M is positive definite, $-\infty$ otherwise. Then replace the objective function above by $-\text{lndet}(B)$; this objective function implicitly requires B to be positive definite. Moreover, we do not have to worry about this non-real-valued function, since if $\{B_k\}$ is a sequence of positive definite matrices converging to a non-positive definite matrix B, then $\ln\det(B_k)$ converges to $-\infty$. It can be shown that $-\text{lndet}$ is a convex function on symmetric matrices.

b) By Cauchy-Schwarz, any x with $\|x\| \leq 1/\|a_j\|$ for all j lies in P, so we can take the ball of radius $\epsilon := 1/\max\{\|a_j\|\}$ as E, i.e., $B = \epsilon^2 I$, $y = 0$ is feasible in the optimization problem above, as is easily checked.

c) If $y = 0$, the constraints become $\sqrt{a_j^T B a_j} \leq 1$ for all j, which is equivalent to $a_j^T B a_j \leq 1$, which is linear in B.

d) If y is a variable, the problem above is not convex in B (think of the 1-dimensional case: B is a number, and $\sqrt{B}$ is not convex). If we square as above, we get $a_j^T B a_j \leq (1 - a_j^T y)^2$, which is convex in B but now not in y. But if we use instead the symmetric square root D of B as our variable, the objective becomes $-\text{lndet}(D^2) = -2\text{lndet}(D)$, and the constraints $a_j^T y + \sqrt{a_j^T D^2 a_j} = a_j^T y + \|Da_j\| \leq 1$ for all j, and this is convex since Da_j is linear in D and the norm is convex!

(A paper by Khachiyan and Todd suggests instead solving a sequence of problems with linear constraints on B and y instead of this one with nonlinear constraints.)