OR 631: Mathematical Programming II. Spring 2014.
Homework Set 3. Due: Tuesday April 15.

1. This question and the next are concerned with central cuts. Suppose we have an ellipsoid $E := E(B, y)$, and we add two cuts symmetrically placed with respect to the center y. Consider

$$\bar{E}_\alpha := \{x \in E : a^T y - \alpha\sqrt{a^T B a} \leq a^T x \leq a^T y + \alpha\sqrt{a^T B a}\}$$

for some nonzero $a \in \Re^n$ and some $0 \leq \alpha \leq 1$.

a) Write the condition for x to lie in $\bar{E}_\alpha$ as two quadratics.

b) By combining these two quadratics suitably, find an ellipsoid $E(B_+, y_+)$ that contains $\bar{E}_\alpha$, depending on a scalar parameter σ.

c) Find the value of σ that minimizes the volume of the resulting ellipsoid as a function of α. Show that for $\alpha = n^{-1/2}$ this ellipsoid is just E, while for α smaller than this it gives an ellipsoid of smaller volume than E. (In fact, this is the minimum-volume ellipsoid among all those containing $\bar{E}_\alpha$, not just those obtained this way.)

2. Consider a centrally symmetric polytope, a bounded polyhedron of the form $P := \{x \in \Re^n : -b \leq Ax \leq b\}$ for some A, b.

a) Show that there is a minimum-volume ellipsoid $E = E(B, y)$ containing P.

b) Show that any such must have $y = 0$, i.e., it must be centrally symmetric also.

c) Show that, if $E(B, 0)$ is a (the) minimum-volume ellipsoid containing P, then $\{n^{-1/2}x : x \in E(B, 0)\}$ is contained in P.

(Hence such polytopes can be rounded to a factor $\sqrt{n}$, not n as in the general case. In fact, this holds for any centrally symmetric convex body.)

3. Suppose that $P := \{x \in \Re^n : A^T x \leq e\}$ is bounded (where e is the vector of ones as usual). Assume that the function $B \to -\ln\det B$ is convex as a function of the entries of the symmetric matrix B.

a) Show how the problem of finding the maximum volume ellipsoid with center y contained in P can be written as an optimization problem with a finite number of constraints. (Argue that the positive semidefiniteness constraint can be eliminated.)

b) Exhibit a feasible solution to this problem.

c) Show that if the center y is restricted to be 0, your optimization problem can be converted to one with linear constraints on B.

d) Now return to the general case, where y is a variable. Try to rewrite the optimization problem with convex constraints (you may want to consider the symmetric square root).