OR 6310: Mathematical Programming II. Spring 2014.
Homework 2 Solutions.

1. Let $P_i \subseteq \mathbf{R}^{d_i}$ be a nonempty polyhedron defined by n_i inequalities, $i = 1, 2$, and let $P := P_1 \times P_2 := \{(x_1; x_2) : x_1 \in P_1, x_2 \in P_2\}$.

a) Show that P is a polyhedron in $\mathbf{R}^d$ defined by n inequalities, with $d = d_1 + d_2$ and $n = n_1 + n_2$, bounded iff both P_1 and P_2 are.

b) Show that, if v_i is a vertex of P_i, $i = 1, 2$, then $(v_1; v_2)$ is a vertex of P. Show that all vertices of P arise in this way.

c) Suppose v_i, v_i' are vertices of P_i, $i = 1, 2$. Show that $[(v_1; v_2), (v_1'; v_2')]$ is an edge of P if $v_1 = v_1'$ and $[v_2, v_2']$ is an edge of P_2, or if $[v_1, v_1']$ is an edge of P_1 and $v_2 = v_2'$. (In fact, all edges of P arise in this way, but you need not prove it; you can assume it for (d).)

d) Show that $\delta(P_1 \times P_2) = \delta(P_1) + \delta(P_2)$.

(This product construction allows you to relate the Hirsch conjecture (of course, now known to be false) for one value of (d, n) to that for other values. Another such construction, the "wedge," converts the polyhedron $Q := \{x \in \mathbf{R}^d : Ax \leq b, a^T x \leq \beta\}$ into the polyhedron $Q' := \{(x; \xi) \in \mathbf{R}^{d+1} : Ax \leq b, a^T x + \xi \leq \beta, -\xi \leq 0\}$. You might want to think of parts (a) – (c) above for Q and Q'. Using these ideas, one can show that the conjecture is true for all values of (d, n) iff it holds for all d and $n = 2d$: this is the so-called d-step conjecture. Similar arguments were used by Santos to modify his "spindle" example in dimension 5 to give a counterexample to the Hirsch conjecture in dimension 43, and to construct counterexamples for all higher dimensions.)

a) Let $P_i = \{x_i \in \mathbf{R}^{d_i} : A_i x_i \leq b_i\}$, $i = 1, 2$, where A_i and b_i have n_i rows. Then $P = \{(x_1; x_2) \in \mathbf{R}^d : A_i x_i \leq b_i, i = 1, 2\}$ and is hence a d-polytope defined by n inequalities. If $\|x_i\| \leq \rho_i$ for all $x_i \in P_i$, for $i = 1, 2$, then $\|x\| \leq \rho_1 + \rho_2$ for all $x \in P$, so P is bounded. Conversely, if say $\|x_1\|$ is unbounded for $x_1 \in P_1$, then choose any fixed $x_2 \in P_2$ and note that $\|(x_1; x_2)\|$ is then unbounded for $(x_1; x_2) \in P$, so P is unbounded.

b) There is an objective function $c_i^T x_i$ that is minimized uniquely over P_i at v_i, $i = 1, 2$. Then $c^T x := c_1^T x_1 + c_2^T x_2$ is minimized uniquely over P at $(v_1; v_2)$, which is therefore a vertex. Conversely, if $c^T x := c_1^T x_1 + c_2^T x_2$ is minimized uniquely over P at $(v_1; v_2)$, then $c_i^T x_i$ is minimized uniquely over P_i at v_i, $i = 1, 2$, so each v_i is a vertex of P_i.

c) Suppose $c_1^T x_i$ is minimized uniquely over P_1 by v_1, and the set of minimizers of $c_2^T x_2$ over P_2 is the line segment joining v_2 and v_2'. Then the set of minimizers of $c^T x := c_1^T x_1 + c_2^T x_2$ is exactly the line segment joining $(v_1; v_2)$ and $(v_1; v_2')$, showing that this is an edge. The same argument works in the other case. To show the converse, note that if $c^T x$ as above defines an edge of P, and neither $c_1^T x_1$ nor $c_2^T x_2$ is uniquely minimized, then the set of minimizers of $c^T x$ contains at least a line segment times a line segment, a contradiction. Suppose therefore without loss of generality that $c_1^T x_1$ is minimized uniquely over P_1 at v_1. Now if $c_2^T x_2$ is minimized over P_2 at more than an edge of P_2, we again get a contradiction.

d) Let $(v_1; v_2)$ and $(w_1; w_2)$ be two vertices of P. There is a path from v_1 to w_1 in P_1 of length at most $\delta(P_1)$, which gives a path of the same length from $(v_1; v_2)$ to $(w_1; v_2)$ in P by just holding the second component fixed. Similarly, there is a path from $(w_1; v_2)$ to $(w_1; w_2)$ of length at most $\delta(P_2)$ holding the first component fixed. Concatenating these two paths gives one of length at most $\delta(P_1) + \delta(P_2)$. To show that the diameter is at least this sum, let v_i and w_i be vertices of P_i a distance $\delta(P_i)$ apart, $i = 1, 2$. Then any path from $(v_1; v_2)$ to $(w_1; w_2)$ gives by projection on its two components a path from v_1 to w_1 and one from v_2 to w_2, and so its total length is at least $\delta(P_1) + \delta(P_2)$.

2. In certain combinatorial optimization problems, the polyhedron defined by certain classes of inequalities is not a 0-1 polytope, but a polytope whose every vertex has components taking on only the values 0, 1/2, or 1. Call such a polytope a (0,1/2,1)-polytope.

Show that every $(0, 1/2, 1)$-polytope in $\mathbf{R}^d$ has diameter at most $2d - 1$. Prove that there is a (0,1/2,1)-polytope in $\mathbf{R}^d$ with diameter $\lfloor 3d/2 \rfloor$ (first consider $d = 1$, 2 and then see if you can blow these examples up to higher dimensions.)

The proof is by induction, being trivial for $d = 1$ (there are seven cases, including the empty polytope). Suppose it is true for $(0, 1/2, 1)$-polytopes of dimension at most $d - 1$, and consider one in $\mathbf{R}^d$, say P, and two vertices of P, say v and w.

a) If $v_1 = w_1 = \alpha$, for $\alpha = 0$ or 1, let $Q := \{x \in P : x_1 = \alpha\}$. Then v and w are vertices of Q (you can prove this using an argument as in Q1, or just assume it). By omitting its first coordinate, Q can be thought of as a polytope in $\mathbf{R}^{d-1}$, so by the inductive hypothesis (since all its vertices are also vertices of P by an argument like that in Q1, so $(0, 1/2, 1)$-valued), there is a path from v to w in Q of length at most $2(d-1)-1$. This is also a path from v to w in P.

b) If v_1 is 0 or 1 (assume wlog the first), and $w_1 = 1/2$, then w is not optimal in $\min\{x_1 : x \in P\}$, so since local optimality implies global optimality for linear optimization in a polytope, there is a vertex z adjacent to w with $z_1 < w_1$. Hence $z_1 = 0$. By a), there is a path in P from v to z of length at most $2(d-1) - 1$, so one from v to w of length at most $2d - 2$.

c) If $v_1 = 0$ and $w_1 = 1$ (or vice versa), then by the same argument as in (b), there is a vertex z of P adjacent to w with $z_1 = 0$ or $1/2$. Then using (a) or (b), there is a path in P from v to z of length at most $2d - 2$, so one from v to w of length at most $2d - 1$.

d) If $v_1 = w_1 = 1/2$, then either $x_1 \geq 1/2$ for all $x \in P$, or $x_1 < 1/2$ for some $x \in P$. In the first subcase, $\{x : x_1 = 1/2\}$ is a supporting hyperplane to P, so $Q := \{x \in P : x_1 = 1/2\}$ is a $(0, 1/2, 1)$-polytope with all its vertices (including v and w) also vertices of P. As in (a), there is a path in Q (hence in P) from v to w of length at most $2(d - 1) - 1$. In the second subcase, as in (b) there is a vertex u adjacent to v with $u_1 = 0$, and a vertex z adjacent to w with $z_1 = 0$. By case (a), there is a path in P from u to z of length at most $2(d - 1) - 1$, so one from v to w of length at most $2d - 1$.

This completes the inductive step and the proof.

The proof of existence of the bad examples is also by induction, starting with $P_1 = [0, 1]$ in $\mathbf{R}^1$ and $P_2 = \{x \in \mathbf{R}^2 : 0 \leq x_j \leq 1, j = 1, 2, 1/2 \leq x_1 + x_2 \leq 3/2\}$ in $\mathbf{R}^2$. These have the required diameters 1 and 3. For larger d, define $P_d = P_2 \times P_{d-2}$. This is a $(0, 1/2, 1)$-polytope in $\mathbf{R}^d$, with diameter $\delta(P_2) + \delta(P_{d-2})$ by Q1, and this is $3 + \lfloor 3(d-2)/2 \rfloor = \lfloor 3d/2 \rfloor$ as desired.

3. a) Consider a polyhedron P and a vertex v of P that uniquely minimizes $c^T x$ over P. Show that $\{x \in P : c^T x \leq \gamma\}$ is bounded for every $\gamma > c^T v$.

b) Klee and Walkup constructed an (unbounded) polyhedron of dimension 4 with 8 facets and diameter $5 > 8 - 4$ violating the Hirsch conjecture, and from this (see Q1) a polyhedron P of dimension $d = 8$ with $n = 16$ facets and diameter $10 = n - d + 2$, also violating the conjecture. Hence P has two vertices a distance 10 apart. Use part (a) to construct a polytope Q of dimension 8 with 17 facets and two vertices v and w of Q so that:

(i) some linear objective function $c^T x$ is minimized uniquely over Q by v; and

(ii) every path from w to v on which $c^T x$ is monotonically decreasing uses at least $10 > 17 - 8$ edges.

(This shows that the “monotonic” Hirsch conjecture is false even for 8-dimensional polytopes. In fact, using projective transformations instead of an extra bounding hyperplane, one can show that it fails even for dimension 4.)

a) If $\{x \in P : c^T x \leq \gamma\}$ is not bounded, it contains a ray $\{u + \lambda w : \lambda \geq 0\}$ for some nonzero w. Because of the extra constraint, $c^T w \leq 0$. If $c^T w < 0$, then $c^T x$ is unbounded below on P, a contradiction, while if $c^T w = 0$, then $v + \lambda w$, $\lambda \geq 0$ shows that $c^T x$ is not uniquely minimized by v, again a contradiction. (Here we have used the representation theorem for polyhedra, which implies that the direction of any ray in Q also is the direction of a ray from every point of Q.)

b) Let v and w be two vertices of P a distance 10 apart. Choose c so that $c^T x$ is uniquely minimized over P by v, and choose γ greater than $c^T u$ for all vertices u of P. Define Q as in (a) using this c and γ. Then $c^T x$ is uniquely minimized over Q by v, so v is a vertex of Q, and if $\tilde{c}^T x$ is uniquely minimized over P by w, it is also uniquely minimized over Q by w, so w is a vertex of Q; similarly, every u that is a vertex of P is also a vertex of Q. Moreover, every vertex u of Q with $c^T x < \gamma$ is a vertex of P: take the same objective that is uniquely minimized over Q at u and use “local implies global” to show that it is also minimized over P at u. Finally, any edge of Q between two vertices u and u' with $c^T x < \gamma$ is also an edge of P, by the same reasoning.

Now consider any path of vertices from w to v on which $c^T x$ is monotonically decreasing. Then every vertex on this path has $c^T x \leq c^T w < \gamma$. It follows that such vertices cannot be on the new facet defined by $c^T x = \gamma$, so they are all vertices of P. Since every path in P from w to v contains at least 10 edges, so does every monotonic path in Q from w to v.