OR 6310: Mathematical Programming II. Spring 2014.
Homework Set 2. Due: Tuesday March 11.

1. Let $P_i \subseteq \mathbf{R}^{d_i}$ be a nonempty polyhedron defined by n_i inequalities, $i = 1, 2$, and let $P := P_1 \times P_2 := \{(x_1; x_2) : x_1 \in P_1, x_2 \in P_2\}$.

a) Show that P is a polyhedron in $\mathbf{R}^d$ defined by n inequalities, with $d = d_1 + d_2$ and $n = n_1 + n_2$, bounded iff both P_1 and P_2 are.

b) Show that, if v_i is a vertex of P_i, $i = 1, 2$, then $(v_1; v_2)$ is a vertex of P. Show that all vertices of P arise in this way.

c) Suppose v_i, v_i' are vertices of P_i, $i = 1, 2$. Show that $[(v_1; v_2), (v_1'; v_2')]$ is an edge of P if $v_1 = v_1'$ and $[v_2, v_2']$ is an edge of P_2, or if $[v_1, v_1']$ is an edge of P_1 and $v_2 = v_2'$. (In fact, all edges of P arise in this way, but you need not prove it; you can assume it for (d).)

d) Show that $\delta(P_1 \times P_2) = \delta(P_1) + \delta(P_2)$.

(This product construction allows you to relate the Hirsch conjecture (of course, now known to be false) for one value of (d, n) to that for other values. Another such construction, the "wedge," converts the polyhedron $Q := \{x \in \mathbf{R}^d : Ax \leq b, a^T x \leq \beta\}$ into the polyhedron $Q' := \{(x; \xi) \in \mathbf{R}^{d+1} : Ax \leq b, a^T x + \xi \leq \beta, -\xi \leq 0\}$. You might want to think of parts (a) – (d) above for Q and Q'. Using these ideas, one can show that the conjecture is true for all values of (d, n) iff it holds for all d and $n = 2d$: this is the so-called d-step conjecture. Similar arguments were used by Santos to modify his "spindle" example in dimension 5 to give a counterexample to the Hirsch conjecture in dimension 43, and to construct counterexamples for all higher dimensions.)

2. In certain combinatorial optimization problems, the polyhedron defined by certain classes of inequalities is not a 0-1 polytope, but a polytope whose every vertex has components taking on only the values 0, 1/2, or 1. Call such a polytope a (0,1/2,1)-polytope.

Show that every (0,1/2,1)-polytope in $\mathbf{R}^d$ has diameter at most $2d - 1$. Prove that there is a (0,1/2,1)-polytope in $\mathbf{R}^d$ with diameter $\lfloor 3d/2 \rfloor$ (first consider $d = 1$, 2 and then see if you can blow these examples up to higher dimensions.)

3. a) Consider a polyhedron P and a vertex v of P that uniquely minimizes $c^T x$ over P. Show that $\{x \in P : c^T x \leq \gamma\}$ is bounded for every $\gamma > c^T v$.

b) Klee and Walkup constructed an (unbounded) polyhedron of dimension 4 with 8 facets and diameter $5 > 8 - 4$ violating the Hirsch conjecture, and from this (see Q1) a polyhedron P of dimension $d = 8$ with $n = 16$ facets and diameter $10 = n - d + 2$, also violating the conjecture. Hence P has two vertices a distance 10 apart. Use part (a) to construct a polytope Q of dimension 8 with 17 facets and two vertices v and w of Q so that:

(i) some linear objective function $c^T x$ is minimized uniquely over Q by v; and

(ii) every path from w to v on which $c^T x$ is monotonically decreasing uses at least $10 > 17 - 8$ edges.

(This shows that the "monotonic" Hirsch conjecture is false even for 8-dimensional polytopes. In fact, using projective transformations instead of an extra bounding hyperplane, one can show that it fails even for dimension 4.)