OR 6310: Mathematical Programming II. Spring 2014.
Homework Set 1 Solutions. Due: Thursday February 20.

1. (KKT solutions and minimizers)

a) Find all KKT solutions ($\bar{x}$'s which with suitable multipliers satisfy the KKT conditions) for $\min\{x_1 - x_1^2 + x_2^2 : -1 \leq x_1 \leq 1\}$, and characterize which are global minimizers, which local but not global minimizers, and which are not even local minimizers.

b) By making a small change to the problem in (a), find a quadratic programming problem where the best KKT solution is not the global minimizer. [This shows that, even if you could "solve" arbitrary LCPs, you can't guarantee finding global minimizers for QPs.]

a) The gradient of the objective is $(1 - 2x_1; 2x_2)$ while those of the constraints are $(-1; 0)$ and $(1; 0)$. The possible sets of active constraints are none, just the first, and just the second, and from examining the KKT conditions we see that x_2 must be zero. So we have $(1/2; 0)$, $(-1; 0)$, and $(1; 0)$ for the three KKT solutions (u equal to $(0; 0)$, $(3; 0)$, and $(0; 1)$ respectively), with objective values $1/4$, -2, and 0 respectively. By examining the univariate quadratic $t - t^2$, it is easy to see that the second is the global minimizer, the third a local but not global minimizer, and the first not even a local minimizer.

b) Motivated by the first solution above, but now working with the second component, change $+x_2^2$ to $-x_2^2$ in the objective. Then it is easy to see that all the KKT solutions above remain KKT solutions, but none of them is even a local minimizer, since small changes in x_2 will improve the objective. In this example, the problem is unbounded; we could make it bounded (no longer quadratic) by using $-x_2^2/(1 + x_2^2)$, but there cannot be a global minimizer (where a constraint qualification held) for then the best KKT point would be a global minimizer.

2. (Symmetric quadratic programming duality)

a) Consider the unconstrained quadratic minimization problem

$$(P): \quad \min_v f(v) := d^T v + \frac{1}{2} v^T H v,$$

where $H = H^T \in \Re^{n \times n}$ is positive semidefinite. Suppose $\nabla f(\bar{v}) = d + H\bar{v} = 0$. Show directly that $\bar{v}$ is a global minimizer for (P), and hence that any two solutions $\bar{v}$ have the same value of f. Show also that, if $d + Hv = 0$ has no solution, then (P) is unbounded below.

b) Now consider the constrained quadratic programming problem

$$(QP): \quad \min_{x,u} c^T x + \frac{1}{2} x^T H x + \frac{1}{2} u^T G u, \quad Ax + Gu \geq b,\ x \geq 0.$$

Here A, H, b, c are as in class and $G = G^T \in \Re^{m \times m}$. Note that if $G = 0$, this is the problem considered in class. Show that (QP) is equivalent to the min-max problem

$$\min_{x,u} \max_{y \geq 0, s \geq 0} L(x, u, y, s),$$

where $L(x, u, y, s)$ is the Lagrangian function

$$L(x, u, y, s) := c^T x + \frac{1}{2} x^T H x + \frac{1}{2} u^T G u + (b - Ax - Gu)^T y + (-x)^T s.$$

Henceforth assume that H and G are positive semidefinite. Next show that the max-min problem

$$\max_{y \geq 0, s \geq 0} \min_{x,u} L(x, u, y, s)$$

is equivalent to the dual problem below (note that this coincides with the dual problem (QD) stated in class if $G = 0$):

$$(QD): \quad \max_{y,v} b^T y - \frac{1}{2} y^T G y - \frac{1}{2} v^T H v, \quad A^T y - Hv \leq c,\ y \geq 0.$$

c) Hence show weak duality directly for this pair of problems.

d) By writing (QD) in the form of (QP), show that the "dual of the dual is the primal."

a) Suppose $d + H\bar{v} = 0$. Then, for any v, $f(v) - f(\bar{v}) = (-H\bar{v})^T v + v^T H v/2 - (-H\bar{v})^T \bar{v} - \bar{v}^T H \bar{v}/2 = (v - \bar{v})^T H (v - \bar{v}) \geq 0$. Now suppose $d + H\bar{v} = 0$ has no solution. Then d does not lie in the range space of H, and since H is psd, this means that d has a component in the null space of H. Choose u with $d^T u < 0$ and $Hu = 0$. Then $f(\lambda u) \to -\infty$ as $\lambda \to \infty$.

b) If $Ax + Gu \geq b$ is false, then $L(x, u, y, s)$ approaches $+\infty$ as $\lambda \to \infty$, for $s = 0$ and $y = \lambda e_i$, some i. Similarly, if $x \geq 0$ is false, then $L(x, u, y, s)$ approaches $+\infty$ as $\lambda \to \infty$, for $y = 0$ and $s = \lambda e_j$, some j. If both these conditions hold, then the maximum is attained for y and s both zero, and equals the objective function of (QP). So the min-max problem is equivalent to (QP). Now consider the max-min problem. Note that $L(x, u, y, s)$ can be written as $x^T H x/2 + (c - A^T y - s)^T x + u^T G u + (-Gy)^T u + b^T y$. By part (a), the first two terms have a minimum in x iff $Hv + c - A^T y - s = 0$ for some v, in which case the minimum is $v^T H v/2 - v^T H v = -v^T H v/2$ for such v. The last two terms have a minimum iff $Gu + (-Gy) = 0$ for some u; but this holds for $u = y$, and then the minimum is $y^T G y/2 - y^T G y = -y^T G y/2$. Thus the max-min problem is equivalent to (QD).

c) Since such inequalities hold in general for min-max and max-min problems, I'll prove it in this generality. Consider an arbitrary function $g(w, z)$ defined for $w \in W$ and $z \in Z$. Then, for any fixed $w' \in W$, and any $z \in Z$, $g(w', z) \geq \min_{w \in W} g(w, z)$. Since this holds for all z, we also have $\max_{z \in Z} g(w', z) \geq \max_{z \in Z} \min_{w \in W} g(w, z)$. And since this holds for every w', we have $\min_{w \in W} \max_{z \in Z} g(w, z) \geq \max_{z \in Z} \min_{w \in W} g(w, z)$.

d) (QD) can be written (changing the sign of the objective function) as

$$(QP') \min_{y,v} (-b)^T y + \frac{1}{2} y^T G y + \frac{1}{2} v^T H v, \quad (-A^T) y + H v \geq -c, y \geq 0,$$

whose dual as above (interchanging A and $-A^T$, b and $-c$, and H and G) is

$$(QD') \max_{x,u} -c^T x - \frac{1}{2} x^T H x - \frac{1}{2} u^T G u, \quad -Ax - Gu \leq -b, x \geq 0,$$

which is equivalent to (QP) after changing the sign of the objective.

3. Our formulation of finding Nash equilibria in a bimatrix game as an LCP does not distinguish one Nash equilibrium from another. Find an LCP so that any nontrivial complementary solution gives a Nash equilibrium where I's expected payoff is at least α and II's expected payoff is at least β. Is it easy to find such Nash equilibria by the same algorithm as discussed in class?

Look at the proof of the theorem in Lecture 3. It actually shows that every Nash equilibrium $(\bar{x}, \bar{y})$ which gives expected payoffs at least α and β to the two players corresponds to a complementary solution $(\bar{u}, \bar{v})$ to the LCP given in the notes with $e_n^T \bar{v} \leq 1/\alpha$ and $e_m^T \bar{u} \leq 1/\beta$. Conversely, any such complementary solution (other than $(0, 0)$) that satisfies the extra constraints corresponds to such a Nash equilibrium.

Thus we want to add these two extra linear constraints to the LCP. If we add two linear equations (with slacks) we also need to add two variables. As in the digression at the end of Lecture 2, we introduce these two variables with zero coefficients in all the rows, so they affect nothing else. Hence our new LCP has

$$M = \begin{bmatrix} 0 & -A & 0 & 0 \\ -B^T & 0 & 0 & 0 \\ -e_m^T & 0 & 0 & 0 \\ 0 & -e_n^T & 0 & 0 \end{bmatrix}, \quad q = \begin{pmatrix} e_m \\ e_n \\ 1/\beta \\ 1/\alpha \end{pmatrix}.$$

Even though this formulation does the job, it is useless in terms of our algorithm. We would again start with $(\bar{u}, \bar{v}) = (0, 0)$ and search G_k for some k. As soon as the slack variable for one of the new constraints hits zero, we would try to increase its complement. But since the added variables have all zero coefficients, the updated coefficients in the current tableau are also zero, and the algorithm would fail on a ray.

4. We made sure that A and B had all positive entries, and then set up a bounded linear system of equations and inequalities to find Nash equilibria of the bimatrix game (A, B).

a) Suppose instead we start by ensuring that all entries of A and B are *negative*, and then consider the LCP defined by

$$M = \begin{pmatrix} 0 & -A \\ -B^T & 0 \end{pmatrix}, \quad q = \begin{pmatrix} -e_m \\ -e_n \end{pmatrix}.$$

Is the corresponding polyhedron bounded? Prove a theorem relating Nash equilibria of the bimatrix game to complementary solutions of this LCP.

b) Try to modify the algorithm we discussed, using k-a.c. basic feasible solutions, to attack the LCP in (a). Show how to initialize it (a couple of special pivots may be required). Do not worry about secondary rays.

a) Since A is negative, we can increase u_1 sufficiently so that t becomes nonnegative, and then increase v_1 sufficiently that s becomes nonnegative. So the polyhedron of feasible solutions to the LCP is nonempty. In fact, we can increase u_1 without bound, so the polyhedron is unbounded. Note that there is now no trivial complementary solution with u and v zero. We have

Theorem 1 *If (x, y) is a Nash equilibrium, then $(u = (-1/x^T By)x, v = (-1/x^T Ay)y)$, with suitable (s, t), gives a complementary solution to the LCP. Conversely, if (s, t, u, v) is a complementary solution to the LCP, then $(x = (1/e_m^T u)u, y = (1/e_n^T v)v)$ is a Nash equilibrium.*

Proof:

Recall that $(x, y) \in X \times Y$ is a Nash equilibrium iff $Ay \leq \alpha e_m$ for some $\alpha \in \Re$, with equality in the ith inequality if $x_i > 0$, and $B^T x \leq \beta e_n$ for some $\beta \in \Re$, with equality in the jth inequality if $y_j > 0$. Here necessarily $\alpha = x^T Ay < 0$, and $\beta = x^T By < 0$. So dividing the inequalities by $-\alpha$ and $-\beta$ and defining u and v as in the theorem, we get $Av \leq -e_m$, with equality in the ith inequality if $u_i > 0$, and $B^T u \leq -e_n$, with equality in the jth inequality if $v_j > 0$. Hence (u, v), with suitable (s, t), gives a complementary solution to the LCP. Conversely, if (s, t, u, v) is a complementary solution to the LCP, then (u, v) satisfies the conditions above. So, by defining x and y as in the theorem, we find that $(x, y) \in X \times Y$ and $Ay \leq (-1/e_n^T v)e_m$, with equality in the ith inequality if $x_i > 0$, and $B^T x \leq (-1/e_m^T u)e_n$, with equality in the jth inequality if $y_j > 0$. Hence (x, y) is a Nash equilibrium as desired. □

b) We can still use the algorithm in class, searching the graph G_k as before, as long as we can find an initial k-a.c. solution and a way to start. Suppose we choose $k = 1$, so we relax the complementarity condition $s_1 u_1 = 0$. Start with the solution with u and v both zero, and s and t basic (but infeasible). Then increase u_1 just enough so that t becomes nonnegative; then t_j is zero for a unique j under a nondegeneracy condition. We now have t_j and v_j both zero. Increase v_j just enough to make s nonnegative, so again under nondegeneracy, a unique s_i is zero. We now have a 1-a.c. solution, with both s_i and u_i zero (unless we were lucky enough to have $i = 1$, in which case we're done). If we increase s_i, its increase is unblocked, as v_j just keeps increasing along with all the other s_h's. So this is the primary ray. Hence we start the algorithm by increasing u_i.

In fact, this algorithm always terminates with a complementary solution. Here is a sketch of why. Each of the (s, v) and (t, u) polyhedra are unbounded, but with a special property of their extreme rays. Indeed, since A and B are negative, these extreme rays correspond to u (or v) being a unit (coordinate) vector, and t (or s) being a positive vector. The extreme rays of the feasible polyhedron are either an extreme ray of the (s, v) polyhedron, with t and u zero, or vice versa. It follows that the only extreme ray consisting of k-a.c. solutions has u_k going to infinity (assuming wlog that $k \leq m$) and all other u_i's and v_j's staying fixed. Thus it must be the primary ray, and so cycling is impossible.