OR 6310: Mathematical Programming II. Spring 2014.
Homework Set 1. Due: Thursday February 20.

1. (KKT solutions and minimizers)

a) Find all KKT solutions ($\bar{x}$'s which with suitable multipliers satisfy the KKT conditions) for $\min\{x_1 - x_1^2 + x_2^2 : -1 \le x_1 \le 1\}$, and characterize which are global minimizers, which local but not global minimizers, and which are not even local minimizers.

b) By making a small change to the problem in (a), find a quadratic programming problem where the best KKT solution is not the global minimizer. [This shows that, even if you could "solve" arbitrary LCPs, you can't guarantee finding global minimizers for QPs.]

2. (Symmetric quadratic programming duality)

a) Consider the unconstrained quadratic minimization problem

$$(P): \quad \min_v f(v) := d^T v + \frac{1}{2} v^T H v,$$

where $H = H^T \in \Re^{n \times n}$ is positive semidefinite. Suppose $\nabla f(\bar{v}) = d + H\bar{v} = 0$. Show directly that $\bar{v}$ is a global minimizer for (P), and hence that any two solutions $\bar{v}$ have the same value of f. Show also that, if $d + Hv = 0$ has no solution, then (P) is unbounded below.

b) Now consider the constrained quadratic programming problem

$$(QP): \quad \min_{x,u} c^T x + \frac{1}{2} x^T H x + \frac{1}{2} u^T G u, \quad Ax + Gu \ge b,\ x \ge 0.$$

Here A, H, b, c are as in class and $G = G^T \in \Re^{m \times m}$. Note that if $G = 0$, this is the problem considered in class. Show that (QP) is equivalent to the min-max problem $\min_{x,u} \max_{y \ge 0, s \ge 0} L(x, u, y, s)$, where $L(x, u, y, s)$ is the Lagrangian function

$$L(x, u, y, s) := c^T x + \frac{1}{2} x^T H x + \frac{1}{2} u^T G u + (b - Ax - Gu)^T y + (-x)^T s.$$

Henceforth assume that H and G are positive semidefinite. Next show that the max-min problem $\max_{y \ge 0, s \ge 0} \min_{x,u} L(x, u, y, s)$ is equivalent to the dual problem below (note that this coincides with the dual problem (QD) stated in class if $G = 0$):

$$(QD): \quad \max_{y,v} b^T y - \frac{1}{2} y^T G y - \frac{1}{2} v^T H v, \quad A^T y - Hv \le c,\ y \ge 0.$$

c) Hence show weak duality directly for this pair of problems.

d) By writing (QD) in the form of (QP), show that the "dual of the dual is the primal."

3. Our formulation of finding Nash equilibria in a bimatrix game as an LCP does not distinguish one Nash equilibrium from another. Find an LCP so that any nontrivial complementary solution gives a Nash equilibrium where I's expected payoff is at least α and II's expected payoff is at least β. Is it easy to find such Nash equilibria by the same algorithm as discussed in class?

4. We made sure that A and B had all positive entries, and then set up a bounded linear system of equations and inequalities to find Nash equilibria of the bimatrix game (A, B).

a) Suppose instead we start by ensuring that all entries of A and B are *negative*, and then consider the LCP defined by

$$M = \begin{pmatrix} 0 & -A \\ -B^T & 0 \end{pmatrix}, \quad q = \begin{pmatrix} -e_m \\ -e_n \end{pmatrix}.$$

Is the corresponding polyhedron bounded? Prove a theorem relating Nash equilibria of the bimatrix game to complementary solutions of this LCP.

b) Try to modify the algorithm we discussed, using k-a.c. basic feasible solutions, to attack the LCP in (a). Show how to initialize it (a couple of special pivots may be required). Do not worry about secondary rays.