An improved Kalai-Kleitman bound for the diameter of a polyhedron

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Abstract

Kalai and Kleitman [1] established the bound $n^{\log(d)+2}$ for the diameter of a d-dimensional polyhedron with n facets. Here we improve the bound slightly to $n^{\log(d)}$.

1 Introduction

A d-polyhedron P is a d-dimensional set in $\mathbb{R}^d$ that is the intersection of a finite number of half-spaces of the form $H := \{x \in \mathbb{R}^d : a^T x \leq \beta\}$. If P can be written as the intersection of n half-spaces H_i, $i = 1, \ldots, n$, but not fewer, we say it has n facets and these facets are the faces $F_i = P \cap H_i$, $i = 1, \ldots, n$, each linearly isomorphic to a $(d-1)$-polyhedron with at most $n-1$ facets. We then call P a (d, n)-polyhedron.

We say $v \in P$ is a *vertex* of P if there is a half-space H with $P \cap H = \{v\}$. Two vertices v and w of P are *adjacent* (and the set $[v, w] := \{(1-\lambda)v + \lambda w : 0 \leq \lambda \leq 1\}$ an *edge* of P) if there is a half-space H with $P \cap H = [v, w]$. A *path of length* k from vertex v to vertex w in P is a sequence $v = v_0, v_1, \ldots, v_k = w$ of vertices with v_{i-1} and v_i adjacent for $i = 1, \ldots, k$. The *distance* from v to w is the length of the shortest such path and is denoted $d_P(v, w)$, and the diameter of P is the largest such distance,

$$\delta(P) := \max\{d_P(v, w) : v \text{ and } w \text{ vertices of } P\}.$$

We define

$$\Delta(d, n) := \max\{\delta(P) : P \text{ a } (d, n)\text{-polyhedron}\}$$

and seek an upper bound on $\Delta(d, n)$. It is not hard to see that $\Delta(d, \cdot)$ is monotonically non-decreasing. Also, the maximum above can be attained by a *simple* polyhedron, one where each vertex lies in exactly d facets. See, e.g., Klee and Kleinschmidt [2] or Ziegler [3]. A related paper, Ziegler [4], gives the history of the Hirsch conjecture on $\Delta_b(d, n)$, defined as above but for bounded polyhedra.

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2 Result

We prove

Theorem 1 *For* $1 \le d \le n$, $\Delta(d,n) \le d^{\log(n)}$.

(All logarithms are to base 2; note that $d^{\log(n)} = n^{\log(d)}$ as both have logarithm $\log(d) \cdot \log(n)$. We use this in the proof below.)

The key lemma is due to Kalai and Kleitman [1], and was used by them to prove the bound $n^{\log(d)+2}$. We give the proof for completeness.

Lemma 1 *For* $2 \le d \le \lfloor n/2 \rfloor$, *where* $\lfloor n/2 \rfloor$ *is the largest integer at most* $n/2$,

$$\Delta(d,n) \le \Delta(d-1,n-1) + 2\Delta(d,\lfloor n/2 \rfloor) + 2.$$

Proof: Let P be a simple (d,n)-polyhedron and v and w two vertices of P with $\delta_P(v,w) = \Delta(d,n)$. We show there is a path in P from v to w of length at most the right-hand side above. If v and w both lie on the same facet, say F, of P, then since F is linearly isomorphic to a $(d-1,m)$-polyhedron with $m \le n-1$, we have $d_P(v,w) \le d_F(v,w) \le \Delta(d-1,m) \le \Delta(d-1,n-1)$ and we are done.

Otherwise, let k_v be the largest k so that there is a set $\mathcal{F}_v$ of at most $\lfloor n/2 \rfloor$ facets with all paths of length k from v meeting only facets in $\mathcal{F}_v$. This exists since all paths of length 0 meet only d facets (those containing v), whereas paths of length $\delta(P)$ can meet all n facets of P. Define k_w and $\mathcal{F}_w$ similarly. We claim that $k_v \le \Delta(d,\lfloor n/2 \rfloor)$ and similarly for k_w. Indeed, let $P_v \supseteq P$ be the (d,m_v)-polyhedron ($m_v = |\mathcal{F}_v| \le \lfloor n/2 \rfloor$) defined by just those linear inequalities corresponding to the facets in $\mathcal{F}_v$. Consider any vertex t of P a distance k_v from v, so there is a shortest path from v to t of length k_v meeting only facets in $\mathcal{F}_v$. But this is also a shortest path in P_v, since if there were a shorter path, it could not be a path in P, and thus must meet a facet not in $\mathcal{F}_v$, a contradiction. So

$$k_v = \delta_{P_v}(v,t) \le \Delta(d,m_v) \le \Delta(d,\lfloor n/2 \rfloor).$$

Now consider the set $\mathcal{G}_v$ of facets that can be reached in at most k_v+1 steps from v, and similarly $\mathcal{G}_w$. Since both these sets contain more than $\lfloor n/2 \rfloor$ facets, there must be a facet, say G, in both of them. Thus there are vertices t and u in G and paths of length at most k_v+1 from v to t and of length at most k_w+1 from w to u. Then

$$\begin{aligned}
\Delta(d,n) &= d_P(v,w) \\
&\le d_P(v,t) + d_G(t,u) + d_P(w,u) \\
&\le k_v + 1 + \Delta(d-1,n-1) + k_w + 1 \\
&\le \Delta(d-1,n-1) + 2\Delta(d,\lfloor n/2 \rfloor) + 2,
\end{aligned}$$

since, as above, G is linearly isomorphic to a $(d-1,m)$-polyhedron with $m \le n-1$. □

Proof of the theorem: This is by induction on $d+n$. First, the right-hand side gives 1 for $d=1$, which is clearly a valid bound, and n for $d=2$ which is an upper bound on the true value of $n-2$.

For $d=3$, if $n<6$ any two vertices lie on a common facet, so their distance is at most $\Delta(2,n-1) = n-3 < n^{\log(3)}$. If $n \ge 6$, we use the lemma to obtain

$$\begin{aligned}
\Delta(3,n) &\le \Delta(2,n-1) + 2 \cdot 3^{\log(\lfloor n/2 \rfloor)} + 2 \\
&\le n - 3 + 2 \cdot 3^{\log(n)-1} + 2 \\
&= n - 1 + \frac{2}{3} \cdot 3^{\log(n)} = n - 1 + \frac{2}{3} \cdot n^{\log(3)}.
\end{aligned}$$

Thus it suffices to show $n - 1 \leq \frac{1}{3} n^{\log(3)}$ for $n \geq 6$, and this can be confirmed by looking at the values at $n = 6$ and the derivatives for $n \geq 6$. (In fact, $\Delta(3, n) = n - 3$; see [2]. We have chosen to give a self-contained argument.)

For $d \geq 4$ and $n < 2d$, the result will follow by induction since any two vertices lie on a common facet giving $\Delta(d, n) \leq \Delta(d-1, n-1)$. For $d = 4$ and $n = 8$, the distance between any two vertices lying on a common facet will be at most $\Delta(3, 7)$ as above, while if v and w lie on disjoint facets, any (bounded) edge from v leads to a vertex u on a common facet with w, so the distance is at most $1 + \Delta(3, 7)$, which again suffices. The only remaining case is $d \geq 4$, $n \geq 9$. For this, $\log(n - 1) \geq 3$, so we have

$$\begin{aligned}
\Delta(d, n) &\leq \Delta(d - 1, n - 1) + 2 \cdot \Delta(d, \lfloor n/2 \rfloor) + 2 \\
&\leq (d - 1)^{\log(n-1)} + 2 \cdot d^{\log(n)-1} + 2 \\
&= \left(\frac{d - 1}{d} \right)^{\log(n-1)} d^{\log(n-1)} + \frac{2}{d} \cdot d^{\log(n)} + 2 \\
&\leq \left(\frac{d - 1}{d} \right)^{3} d^{\log(n)} + \frac{2}{d} \cdot d^{\log(n)} + 2 \\
&\leq d^{\log(n)} - \frac{3}{d} \cdot d^{\log(n)} + \frac{3}{d^2} \cdot d^{\log(n)} - \frac{1}{d^3} \cdot d^{\log(n)} + \frac{2}{d} \cdot d^{\log(n)} + 2 \\
&= d^{\log(n)} - \frac{1}{d} \cdot d^{\log(n)} + \frac{3}{d^2} \cdot d^{\log(n)} - \frac{1}{d^3} \cdot d^{\log(n)} + 2 \\
&\leq d^{\log(n)} - \frac{1}{d} \cdot d^{\log(n)} + \frac{3}{4d} \cdot d^{\log(n)} - \frac{1}{d^3} \cdot d^{\log(n)} + 2 \\
&\leq d^{\log(n)} - \frac{1}{4d} \cdot d^{\log(n)} - \frac{1}{d^3} \cdot d^{\log(n)} + 2 \\
&\leq d^{\log(n)},
\end{aligned}$$

since each of the subtracted terms is at least one. This completes the proof. □

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References

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