Computation, Multiplicity, and Comparative Statics of Cournot Equilibria in Integers

Michael J. Todd [*]

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Abstract

We give an efficient algorithm for computing a Cournot equilibrium when the producers are confined to integers, the inverse demand function is linear, and costs are quadratic. The method also establishes existence constructively. We use our characterization to discuss the multiplicity of integer Cournot equilibria and their relationship to the real Cournot equilibrium.

1 Introduction

Consider n producers who produce a single good. If each producer i produces q_i, we assume the price at which the good can be sold is

$$p = (a - bQ)_+, \quad \text{where } Q := \sum_{i=1}^{n} q_i,$$

here and below, denotes the total production, a and b are positive, and z_+ denotes the positive part $\max\{0, z\}$ of z. Producer i also faces a production cost of $c_i q_i + d_i q_i^2$, with c_i and d_i nonnegative with positive sum. Then if the production vector is $q = (q_i)$, producer i makes profit

$$\pi_i(q) := (a - bQ)_+ q_i - c_i q_i - d_i q_i^2.$$

With a slight abuse of notation, we also write $\pi_i(q)$ as $\pi_i(q_i, q_{-i})$ or $\pi_i(q_i, Q_{-i})$, where $q_{-i} = (q_j)_{j \neq i}$ and $Q_{-i} := \sum_{j \neq i} q_j$, to highlight its dependence on producer i's decision as well as those of the other producers. Note that

$$\pi_i(q_i, Q_{-i}) = (a - bq_i - bQ_{-i})_+ q_i - c_i q_i - d_i q_i^2.$$

Our main interest is in the case that each producer chooses her decision q_i from the nonnegative integers, $\mathbb{Z}_+$, but we also consider the simpler case where q_i is chosen from the nonnegative reals, $\mathbb{R}_+$. In either case, a Cournot equilibrium is a production profile where each producer i chooses her best response to the decisions of the other producers:

[*]School of Operations Research and Information Engineering, Cornell University, Ithaca, NY 14853, USA. E-mail `mjt7@cornell.edu`.

Definition 1 *$q \in \mathbb{R}^n_+$ ($q \in \mathbb{Z}^n_+$) is a real (integer) Cournot equilibrium for $\{\pi_i\}$ if for each i*

$$q_i \text{ maximizes } \pi_i(\cdot, Q_{-i}) \text{ over } \mathbb{R}_+(\mathbb{Z}_+).$$

Cournot equilibria, with the related Bertrand equilibria (where the producers choose prices, not quantities) and Stackelberg (leader-follower) variants of these, form a standard topic in oligopoly theory: see, for example Mas-Colell, Whinston, and Green [4], pp. 389–394, or Varian [11], pp. 285–291. Kreps and Scheinkman [2] show how a Cournot equilibrium arises from a two-stage game: first the producers choose capacities, and then, knowing these capacities, each chooses a price as in the Bertrand model. Integer or discrete versions are much less studied, although existence is proved in for example Kukushkin [3] using Tarski's fixed-point theorem, and Dubey, Haimanko, and Zapechelnyuk [1] using a potential game, following a less general treatment of Shapley [7].

This paper is concerned with the characterization and efficient computation of real and integer Cournot equilibria, and as a byproduct, their existence, in the case of a linear inverse demand function and convex quadratic costs as above. While this case is quite restrictive, we note that it is prototypical, and that our technique can be used in an iterative scheme where at each step, the inverse demand function and the cost functions are approximated by a linear function, respectively quadratic functions, at the current iterates. While several economies, e.g., electricity markets, exhibit indivisibilities in production, so that discrete choices arise naturally in equilibrium models, it is important to note that our analysis requires that all producers can only choose an integer multiple of some common unit. A similar assumption is used by Kukushkin [3].

Our characterization of equilibria also allows us to discuss the multiplicity of equilibria. As is well-known, under our assumptions real Cournot equilibria are unique. We show that, while there may be many integer Cournot equilibria, the total production is either unique or one of two adjacent integers. We also show that the distance of the total production in an integer Cournot equilibrium from that in the real equilibrium is at most $n/2$, and that this bound is tight. However, this does not mean that each producer's choice is within $1/2$ of that in the real equilibrium; we have examples where this is about $n/4$, and a proof that it is at most $n/2+1$. These comparative statics results give another justification for studying integer Cournot equilibria. Most goods are produced in discrete quantities, but the discretization is fine enough that it is assumed that real equilibria will give good approximations. Our results, however, show that the relative error in the quantity chosen by an individual producer in the real Cournot equilibrium as opposed to an integer equilibrium can be large whenever the number of agents in the economy is comparable to this quantity. Since integer Cournot equilibria can be computed efficiently, the justification for using the real approximation is rather weak.

The existence of equilibria is usually established by showing that the game above is a potential game, i.e., there is a function of q whose maximizers yield Cournot equilibria. This can be done in both the real case (Monderer and Shapley [5]) and the integer case (Shapley [7]). There are algorithms to compute the maximizers, but they are not nearly as efficient as those we will propose, especially in the integer case.

Computation of Cournot equilibria in much more general settings is discussed for example in Sherali, Soyster, and Murphy [8] and Thorlund-Petersen [10]. Our basic scheme of performing a binary search on the total production can be applied in much more general frameworks in the integer case, but the algorithms will be far less efficient.

In the next section we give useful characterizations of equilibria, constructive proofs of existence, and efficient algorithms to compute equilibria in both the real and integer cases. Surprisingly, the complexity of these algorithms is of the same order, $O(n \log n)$, for the two cases.

In Section 3 we discuss multiplicity of equilibria in the integer case and prove our comparative statics results on the distance of integer equlibria from real equilibria. We also give some examples to illustrate the algorithms, the multiplicity of integer Cournot equilibria and the differences between integer and real equilibria.

2 Existence and Computation

First we simplify the payoff functions. While negative prices have no interpretation in our setting, we can define payoff functions as if the price were given by $a - bQ$ instead of its positive part. Consider (again with a slight abuse of notation)

$$\hat{\pi}_i(q_i, q_{-i}) := \hat{\pi}_i(q_i, Q_{-i}) := (a - bq_i - bQ_{-i})q_i - c_i q_i - d_i q_i^2.$$

Proposition 1 *The real (integer) Cournot equilibria for $\{\hat{\pi}_i\}$ coincide with those for $\{\pi_i\}$.*

Proof: Note that in any real or integer Cournot equilibrium q for either set of payoff functions, every producer gets a nonnegative payoff, since producing nothing is always an alternative. But then we cannot have $a - bQ$ nonpositive, for then any producer with positive q_i (and there must be one) gets negative payoff. Since the two payoff functions only differ for $a - bQ$ negative, and no producer has an incentive to move to such a point, an equilibrium for one set of payoff functions is also one for the other. $\square$

We can now use the payoff functions $\{\hat{\pi}_i\}$ in our equilibrium arguments.

Let us write

$$e_i := a - c_i$$

for each i for convenience. We can assume that e_i is positive for all i, since any producer with nonpositive e_i would choose $q_i = 0$ whatever the others do.

The strictly concave quadratic

$$\hat{\pi}_i(q_i, Q_{-i}) = (e_i - bQ_{-i})q_i - (b + d_i)q_i^2,$$

for any nonnegative Q_{-i}, is maximized in q_i over the reals uniquely at

$$\frac{e_i - bQ_{-i}}{2(b + d_i)},$$

and over the nonnegative reals at

$$\bar{q}_i^* := \frac{(e_i - bQ_{-i})_+}{2(b+d_i)}, \tag{1}$$

as is easily seen from the first-order conditions or directly.

Since quadratics are symmetric about their maximizers, the maximum over the integers is attained at any integer in $[\bar{q}_i^-, \bar{q}_i^+]$, where

$$\bar{q}_i^\pm \quad := \quad \left(\tfrac{e_i - bQ_{-i}}{2(b+d_i)} \pm \tfrac{1}{2}\right)_+ = \tfrac{(e_i \pm (b+d_i) - bQ_{-i})_+}{2(b+d_i)}. \tag{2}$$

In both cases above, the optimal value or bounds for q_i are given in terms of Q_{-i}, but it would be much more useful if they were defined (in a somewhat circular way) as functions of the total production Q. Such functions were first considered by Szidarovszky and Yakowitz [9] (see also Novshek [6], who called them "backward reaction mappings"). For this reformulation the following result is helpful:

Lemma 1 *For real λ and $0 < \mu < 1$,*

$$x\,R\,(\lambda - \mu y)_+ \text{ iff } x\,R\left(\frac{\lambda}{1-\mu} - \frac{\mu}{1-\mu}(y+x)\right)_+,$$

where R stands for less than or equal to, greater than or equal to, or equal to.

Proof: We prove the lemma when R is "$\leq$" and "$\geq$"; the third case then follows. For the first case, the left-hand side is true iff $x \leq 0$ or $0 < x \leq \lambda - \mu y$, while the right hand-side is true iff

$$x \leq 0 \text{ or } 0 < x \leq \frac{\lambda}{1-\mu} - \frac{\mu}{1-\mu}(y+x).$$

It is easy to see that these are equivalent.

For the second case, the left-hand side is true iff $x \geq 0$ and $x \geq \lambda - \mu y$, while the right hand-side is true iff

$$x \geq 0 \text{ and } x \geq \frac{\lambda}{1-\mu} - \frac{\mu}{1-\mu}(y+x),$$

and once again these are easily seen to be equivalent. □

We now apply the lemma to the conditions for optimal q_i's above.

Theorem 1 *a) A production profile $q^* \in \mathbb{R}_+^n$ is a real Cournot equilibrium iff*

$$q_i^* = q_i^*(Q^*) := \frac{(e_i - bQ^*)_+}{b + 2d_i}$$

for each i.

b) A production profile $\bar{q} \in \mathbb{Z}_+^n$ is an integer Cournot equilibrium iff $\bar{q}_i \in [q_i^-(\bar{Q}), q_i^+(\bar{Q})]$, where

$$q_i^\pm(Q) \quad := \quad \frac{(e_i \pm (b+d_i) - bQ)_+}{b+2d_i},$$

for each i.

Proof: For part (a), we apply the lemma to (1) with R "$=$" and $\lambda = e_i/[2(b+d_i)]$, $\mu = b/[2(b+d_i)]$.

For part (b), we apply the lemma to (2) with R "$\geq$" and $\lambda = [e_i - b - d_i]/[2(b+d_i)]$, $\mu = b/[2(b+d_i)]$, and with R "$\leq$" and $\lambda = [e_i + b + d_i]/[2(b+d_i)]$, $\mu = b/[2(b+d_i)]$. □

We can now investigate efficient algorithms to compute Cournot equilibria. In the real case, it suffices to find $Q^* \in \mathbb{R}_+$ satisfying

$$Q = \phi(Q) := \sum_i \frac{(e_i - bQ)_+}{b + 2d_i}.$$

This provides a constructive proof of existence, since ϕ is continuous and nonincreasing, positive at zero, and zero for large enough Q, and hence has a unique fixed point Q^* by the intermediate-value theorem. It seems that to find the fixed point we need to do a binary search on Q to determine for which indices the numerator is positive, but if the e_i's are sorted (which takes $O(n \log n)$ time), this can be avoided by computing the desired sums incrementally. Let us assume the e_i's are in nonincreasing order. Then for some $1 \leq j \leq n$,

$$\begin{aligned} Q^* &= \sum_{i=1}^{j} \frac{e_i - bQ^*}{b + 2d_i} \\ &= \frac{\sum_{i=1}^{j} e_i/(b+2d_i)}{1 + \sum_{i=1}^{j} b/(b+2d_i)} \end{aligned}$$

and $e_j/b \geq Q^* \geq e_{j+1}/b$, where we take $e_{n+1} := 0$. This gives our algorithm, where we successively compute the numerator (ν) and the denominator (δ), terminating when the ratio exceeds e_{j+1}/b.

Algorithm 1 *(Computing a real Cournot equilibrium)*

Initialize $\nu = 0$ and $\delta = 1$.

For $j = 1, 2, \ldots, n$,

$\nu \leftarrow \nu + \frac{e_j}{b+2d_j}$, $\delta \leftarrow \delta + \frac{b}{b+2d_j}$.

If $\frac{\nu}{\delta} \geq \frac{e_{j+1}}{b}$, set $Q^* = \frac{\nu}{\delta}$ and $q_i^* = q_i^*(Q^*)$, $i = 1, \ldots, n$, and STOP.

Next j.

It is easy to see inductively that $\nu/\delta \leq e_j/b$. Indeed, the previous ν/δ was less than e_j/b either because it is initially zero or because the test at the previous j failed, and the current value is a convex combination of the old value and e_j/b. Hence we obtain

Theorem 2 *Algorithm 1 finds a real Cournot equilibrium in $O(n \log n)$ time (or $O(n)$ time if the e_i's are already sorted).*

The situation is more complicated in the integer case. We can proceed as in the real case to obtain bounds on Q for any integer Cournot equilibrium. Indeed, let us define

$$\phi^{\pm}(Q) := \sum_i \frac{(e_i \pm (b + d_i) - bQ)_+}{b + 2d_i}$$

and note that both these functions are nonincreasing in Q, and hence have unique fixed points Q^- and Q^+. If $Q < Q^-$, we have $Q < \sum_i q_i^-(Q)$, and so Q cannot be the total production of any integer Cournot equilibrium. Similarly, Q cannot be the total production of any integer Cournot equilibrium if $Q > Q^+$. This gives bounds on the total production of any integer Cournot equilibrium, which can be computed as Q^* above. So, if $(ebd^-_{[i]})$ denotes the components of $(e_j - b - d_j)$ arranged in nonincreasing order, and $(d_{[i]})$ denotes the components of (d_j) in the same order, then for some $j = 1, \ldots, n$,

$$\begin{aligned} Q^- &= \sum_{i=1}^{j} \frac{ebd^-_{[i]} - bQ^-}{b + 2d_{[i]}}, \text{ so} \\ Q^- &= \frac{\sum_{i=1}^{j} ebd^-_{[i]}/(b + 2d_{[i]})}{1 + \sum_{i=1}^{j} b/(b + 2d_{[i]})} \end{aligned} \tag{3}$$

where $ebd^-_{[j]}/b \geq Q^- \geq ebd^-_{[j+1]}/b$ and again we take $ebd^-_{[n+1]} := 0$. Similarly, if $(ebd^+_{\{i\}})$ denotes the components of $(e_j + b + d_j)$ arranged in nonincreasing order, and $(d_{\{i\}})$ denotes the components of (d_j) in the same order, then for some $j = 1, \ldots, n$,

$$\begin{aligned} Q^+ &= \sum_{i=1}^{j} \frac{ebd^+_{\{i\}} - bQ^+}{b + 2d_{\{i\}}}, \text{ so} \\ Q^+ &= \frac{\sum_{i=1}^{j} ebd^+_{\{i\}}/(b + 2d_{\{i\}})}{1 + \sum_{i=1}^{j} b/(b + 2d_{\{i\}})} \end{aligned} \tag{4}$$

where $ebd^+_{\{j\}}/b \geq Q^+ \geq ebd^+_{\{j+1\}}/b$ with $ebd^+_{[n+1]} := 0$.

While $Q^\pm$ exist, unlike in the real case we do not have a proof of existence or an algorithm for integer equilibria directly. The difficulties are several. First, these conditions are necessary but not sufficient, since for ease of analysis we have neglected the requirement that each q_i be integer. Second, while both $\sum_i \lceil q_i^-(Q) \rceil$ and $\sum_i \lfloor q_i^+(Q) \rfloor$ are nonincreasing in Q, and it is not hard to see that this range always includes an integer (see the proof of Theorem 3 below), it is *a priori* possible that the range jumps over Q as it moves from one integer value to the next. Hence existence does not follow directly. Any algorithm to compute integer Cournot equilibria also has to take into account the integer restrictions.

We deal with these problems sequentially as follows. First, we use the characterization of integer equilibria directly to prove existence. Next, we use the necessary bounds above to restrict the range of possible Q's. We show how they may be computed efficiently and prove their usefulness by bounding their distance from the total production in a real equilibrium. Then we perform a binary search over this range, using the correct bounds with integer parts for the individual q_i's to find an integer Cournot equilibrium.

We now show directly that an integer Cournot equilibrium exists. For this, we show that iteratively trying $Q = 0, 1, \ldots$ will give a value for which the necessary and sufficient conditions of Theorem 1 are satisfied.

Theorem 3 *For some nonnegative integer Q,*

$$\sum_i \lceil q_i^-(Q) \rceil \leq Q \leq \sum_i \lfloor q_i^+(Q) \rfloor,$$

implying that an integer Cournot equilibrium exists.

Proof: Note first that both $\lceil q_i^-(Q) \rceil$ and $\lfloor q_i^+(Q) \rfloor$ are nonincreasing in Q, and also that there is always an integer between them. This is trivial if $\lceil q_i^-(Q) \rceil = 0$, and if not, $q_i^+(Q) = q_i^-(Q) + 2(b+d_i)/(b+2d_i) > q_i^-(Q)+1$, from which the claim follows.

Next, for $Q = 0$, $\sum_i \lceil q_i^-(Q) \rceil$ is either zero, in which case this Q satisfies our inequalities, or positive. In the latter case, we know that $\sum_i \lceil q_i^-(Q) \rceil$ is zero for $Q \geq a/b$. Hence there is a first Q, say $\bar{Q}$, for which

$$\sum_i \lceil q_i^-(\bar{Q}) \rceil \leq \bar{Q}, \tag{5}$$

so that

$$\sum_i \lceil q_i^-(\bar{Q}-1) \rceil > \bar{Q} - 1.$$

Now for any i, if $q_i^-(\bar{Q}-1)$ is positive,

$$q_i^-(\bar{Q}-1) + 1 = \frac{e_i - b - d_i - b(\bar{Q}-1)}{b + 2d_i} + 1 = \frac{e_i + b + d_i - b\bar{Q}}{b + 2d_i} = q_i^+(\bar{Q}),$$

and so $\lceil q_i^-(\bar{Q}-1) \rceil \leq q_i^-(\bar{Q}-1) + 1 = q_i^+(\bar{Q})$, whence $\lceil q_i^-(\bar{Q}-1) \rceil \leq \lfloor q_i^+(\bar{Q}) \rfloor$. On the other hand, if $q_i^-(\bar{Q}-1)$ is zero, then this inequality holds trivially.

Hence

$$\sum_i \lfloor q_i^+(\bar{Q}) \rfloor \geq \sum_i \lceil q_i^-(\bar{Q}-1) \rceil \geq \bar{Q}. \tag{6}$$

But now (5) and (6) imply that the inequalities in the theorem are satisfied, and hence an integer Cournot equilibrium q exists. Indeed, we can start with $q_i = \lceil q_i^-(\bar{Q}) \rceil$ for all i and then increase q_i's within their range until a sum of $\bar{Q}$ is achieved.

□

The algorithm below computes lower and upper bounds, $Q^\pm$, on Q in any integer Cournot equilibrium as we computed Q^* above:

Algorithm 2 *(Computing bounds for an integer Cournot equilibrium)*

Initialize $\nu = 0$ *and* $\delta = 1$.

For $j = 1, 2, \ldots, n$,

$\nu \leftarrow \nu + \frac{ebd_{[j]}^-}{b+2d_{[j]}}$, $\delta \leftarrow \delta + \frac{b}{b+2d_{[j]}}$.

If $\frac{\nu}{\delta} \geq \frac{ebd_{[j+1]}^-}{b}$, *set* $Q^- = \frac{\nu}{\delta}$ *and BREAK.*

Next j.

Initialize $\nu = 0$ *and* $\delta = 1$.

For $j = 1, 2, \ldots, n$,

$\nu \leftarrow \nu + \frac{ebd_{\{j\}}^+}{b+2d_{\{j\}}}$, $\delta \leftarrow \delta + \frac{b}{b+2d_{\{j\}}}$.

If $\frac{\nu}{\delta} \geq \frac{ebd_{\{j+1\}}^+}{b}$, *set* $Q^+ = \frac{\nu}{\delta}$ *and STOP.*

Next j.

This algorithm computes the values $Q^{\pm}$ correctly for exactly the same reason that Algorithm 1 computes Q^* correctly. We showed above that these quantities provide valid bounds on the total production for any integer Cournot equilibrium. Hence Algorithm 2 provides correct bounds.

Next we prove the usefulness of these bounds by showing

Theorem 4 *If Q^* denotes the total production for the real Cournot equilibrium,*

$$Q^- \geq Q^* - \frac{n}{2} \text{ and } Q^+ \leq Q^* + \frac{n}{2}.$$

Hence the total production in an integer Cournot equilibrium differs from that in the real equilibrium by at most $n/2$. Similarly, each producer's production in an integer Cournot equilibrium differs from her production in the real equilibrium by at most $n/2 + 1$.

Proof: We have

$$\begin{aligned} q_i^-\left(Q^* - \frac{1}{2}\right) &= \frac{(e_i - b/2 - d_i - bQ^*)_+}{b + 2d_i} \\ &= \left(\frac{e_i - bQ^*}{b + 2d_i} - \frac{1}{2}\right)_+ \\ &\geq \frac{(e_i - bQ^*)_+}{b + 2d_i} - \frac{1}{2}, \end{aligned}$$

and so $\phi^-(Q^* - 1/2) \geq Q^* - n/2$, which shows that $Q^- \geq Q^* - n/2$. In the same way, we can show that $Q^+ \leq Q^* + n/2$.

Similarly, if Q is the total production in an integer Cournot equilibrium, for each i we have

$$\begin{aligned} q_i^-(Q) &\geq q_i^-(Q^+) \\ &= \frac{(e_i - b - d_i - bQ^+)_+}{b + 2d_i} \\ &\geq \frac{(e_i - b - d_i - bQ^* - bn/2)_+}{b + 2d_i} \\ &\geq \frac{(e_i - bQ^*)_+}{b + 2d_i} - 1 - \frac{n}{2} = q_i^* - \frac{n}{2} - 1, \end{aligned}$$

and similarly $q_i^+(Q) \leq q_i^* + n/2 + 1$, which shows the last part. $\square$

We now provide a simple binary search to obtain an integer Cournot equilibrium based on Theorem 1.

Algorithm 3 *(Computing an integer Cournot equilibrium)*

Use Algorithm 2 to find Q^- and Q^+.
Set $l := \lceil Q^- \rceil$, $u := \lfloor Q^+ \rfloor$.
While $u - l > 0$,
  Set $Q = \lceil (l + u)/2 \rceil$.
  If $\sum_i \lceil q_i^-(Q) \rceil > Q$,
    $l := Q + 1$;
  Elseif $\sum_i \lfloor q_i^+(Q) \rfloor < Q$,
    $u := Q - 1$;
  Else $l := Q$ and BREAK.
Endwhile
Set $\bar{Q} := l$ and $\bar{q}_i$ between $\lceil q_i^-(\bar{Q}) \rceil$ and $\lfloor q_i^+(\bar{Q}) \rfloor$ so that $\sum_i \bar{q}_i = \bar{Q}$.

Theorem 5 *Algorithm 3 correctly obtains an integer Cournot equilibrium in $O(n \log n)$ time.*

Proof: By Theorem 3, there exists an integer Cournot equilibrium $\bar{q}$, and with $\bar{Q} := \sum_i \bar{q}_i$, we then have $\sum_i \lceil q_i^-(\bar{Q}) \rceil \leq \bar{Q} \leq \sum_i \lfloor q_i^+(\bar{Q}) \rfloor$. Since these two sums are nonincreasing functions of Q, it follows that the binary search will successfully find such a $\bar{Q}$ given that the initial bounds are valid. But this was established below Algorithm 2.

For the time complexity, we require $O(n \log n)$ time to order the $e_i - b - d_i$'s and $e_i + b + d_i$'s, and then $O(n)$ time for Algorithm 2. Each step of the binary search takes $O(n)$ time, and there are at most $\log n$ steps since Theorem 4 proves $Q^+ - Q^- \leq n$. $\square$

In the next section we show that the total production in an integer Cournot equilibrium can be at most two consecutive integers. Thus having obtained one equilibrium by the algorithm above, we can check at most two other values of Q and hence obtain, at least up to the distribution of the total production between the individual producers satisfying their bounds, all integer Cournot equilibria in the same time complexity.

3 Examples, Multiplicity, and Comparative Statics

We first give some examples to show how the algorithm works and to illustrate some surprising features of integer Cournot equilibria.

Examples.

Suppose $a = 5$, $b = 1$, and there are ten producers with costs given by

$$c = (0,0,0,1,1,1,1,2,2,2), \qquad d = (1,1,1,1,1,1,1,0,0,0).$$

Then

$$e = (5,5,5,4,4,4,4,3,3,3).$$

Note that these are in nonincreasing order. Also,

$$(b + 2d_i) = (3,3,3,3,3,3,3,1,1,1).$$

For the real case, Algorithm 1 increases j to 7, and then $\nu = 31/3$ and $\delta = 10/3$, so $\nu/\delta = 31/10 > e_8$ and the loop ends. Thus $Q^* = 31/10$, and we have found the real Cournot equilibrium

$$q^* = q^*(Q^*) = (\frac{19}{30}, \frac{19}{30}, \frac{19}{30}, \frac{3}{10}, \frac{3}{10}, \frac{3}{10}, \frac{3}{10}, 0, 0, 0).$$

It is easy to check that this is indeed an equilibrium, either from Theorem 1 or directly from (1).

In the integer case, we first compute

$$(e_i - b - d_i) = (3,3,3,2,2,2,2,2,2,2)$$

and

$$(e_i + b + d_i) = (7,7,7,6,6,6,6,4,4,4).$$

Note that these are both in nonincreasing order. We first apply Algorithm 2. For the lower bound, j increases to 10, and then $\nu = 35/3$ and $\delta = 19/3$, so $Q^- = 35/19$ with ceiling 2. For the upper bound, j increases to 7, with $\nu = 45/3$ and $\delta = 10/3$, so $\nu/\delta = 9/2 > ebd_8^+$. So the loop ends, and $Q^+ = 9/2$ with floor 4.

Now we do binary search. The first trial Q is 3. We find

$$q^-(Q) = (0,0,0,0,0,0,0,0,0,0), \qquad q^+(Q) = (\frac{4}{3}, \frac{4}{3}, \frac{4}{3}, 1,1,1,1,1,1,1).$$

So the lower bounds are all zero, and the upper bounds all 1. It follows that any $(0,1)$ vector with three ones is an integer Cournot equilibrium. In particular, $(0,0,0,0,0,0,0,1,1,1)$ is an integer equilibrium, whereas the last three producers always produce zero in the (unique) real Cournot equilibrium. Again, it is easy to confirm that this is indeed an integer Cournot equilibrium, either from Theorem 1 or directly from (2).

We could also try Q equal to 2 or 4, but in both cases the inequalities in Theorem 3 fail, so these do not lead to equilibria.

To show the value of obtaining bounds in Algorithm 2, note that if we keep the data above except that we change a to 50, then the algorithm gives bounds of 40 and 42, so that only at most two values of Q need to be checked (and 41 and 42 both turn out to yield equilibria, whereas 40 does not); by contrast, a binary search on the range [0,50] would require up to six tests.

A last point illustrated by this example is the following: since the payoff functions are strictly concave quadratics, one might expect that the output chosen by a producer in a Cournot equilibrium would either be a unique integer, or perhaps one of two consecutive integers. However, in this example, for $Q = 41$, we obtain

$$q^-(Q) = (3,3,3,2,2,2,2,6,6,6), \qquad q^+(Q) = (3,3,3,3,3,3,3,8,8,8).$$

Hence there are equilibria where the last three producers choose either 6, or 7, or 8, for example for producer 8, $q = (3,3,3,3,3,3,3,6,8,6)$, or $q = (3,3,3,3,3,3,3,7,7,6)$, or $q = (3,3,3,3,3,3,3,8,6,6)$. This example also shows that in an integer Cournot equilibrium, a producer's choice may differ by more than one from her choice in the real equilibrium; for this example, the last three producers choose $q_i = 7\frac{1}{19}$ in the real equilibrium.

Similarly, the total production in an integer Cournot equilibrium can differ by much more than one from that for a real Cournot equilibrium. If $n = 100$, $a = 1101$, $b = 1$, c is a vector of ones, and d is 999.5 times c, the total production in the real Cournot equilibrium is a little over 52 but the only integer Cournot equilibria has total production 100. If we change a to 1001, then the total production in the real Cournot equilibrium is a little under 48, but the only integer Cournot equilibrium has total production 0.

We have seen above that many integer Cournot equilibria may exist, but in these examples, only at most two different values for the total production are possible, although these may be far from the total production in the real Cournot equilibrium. We now show this in general.

Theorem 6 *There may be exponentially many integer Cournot equilibria. However, at most two different total production quantities are possible, and these will be adjacent integers. For any positive ϵ, the total production in an integer Cournot equilibrium can be more than $n/2 - \epsilon$ from that in the real equilibrium, and an individual producer's choice can be more than $(n-1)/4 - \epsilon$ from her choice in the real equilibrium.*

Proof: First we provide an example showing that there may be many integer Cournot equilibria. Suppose there are n producers (n even), each with $c_i = 1$ and $d_i = 0$. Suppose $a = n/2 + 2$ and $b = 1$. Then for $Q = n/2$, we find $q_i^{\pm}(Q) = 1 \pm 1$, so values $0, 1$, and 2 are possible. Hence there are $\binom{n}{n/2}$ equilibria where $n/2$ producers choose $q_i = 1$ and the others choose $q_i = 0$. There are further equilibria where up to $n/4$ producers choose $q_i = 2$, but we already have exponentially many. In addition, for $Q = n/2 + 1$, we find $q_i^{\pm}(Q) = (0 \pm 1)_+$, so values 0 and 1 are possible. Hence there are another $\binom{n}{n/2+1}$ equilibria where $n/2 + 1$ producers choose $q_i = 1$ and the rest choose $q_i = 0$.

Next we prove that only at most two adjacent integer values are possible for the total production in an integer Cournot equilibrium. Let $q \in \mathbb{Z}^n$ be an integer Cournot equilibrium with the smallest total production Q. Let $j \geq 2$. We show that no integer Cournot equilibrium can have total production $Q + j$. For this, we use the

Claim. For all i,

$$\lfloor q_i^+(Q+j) \rfloor \leq \lceil q_i^-(Q) \rceil.$$

Assuming the claim, we see that if $\bar{q}$ were an integer Cournot equilibrium with total production $Q + j$, then we would have

$$Q + j = \sum_i \bar{q}_i \leq \sum_i \lfloor q_i^+(Q+j) \rfloor \leq \sum_i \lceil q_i^-(Q) \rceil \leq Q,$$

a contradiction. To prove the claim, we consider three cases:

(i) Suppose $\lceil q_i^-(Q) \rceil = \lfloor q_i^+(Q) \rfloor$. Then since $\lfloor q_i^+(Q+j) \rfloor \leq \lfloor q_i^+(Q) \rfloor$, the claim is established.

(ii) Suppose $d_i = 0$. Then

$$q_i^+(Q+j) = \frac{(e_i + b - b(Q+j))_+}{b} \leq \frac{(e_i - b - bQ)_+}{b} = q_i^-(Q),$$

again establishing the claim.

(iii) Suppose $\lceil q_i^-(Q) \rceil < \lfloor q_i^+(Q) \rfloor$ and $d_i > 0$. Then

$$\begin{aligned} q_i^+(Q) - q_i^-(Q) &= \frac{(e_i - bQ + b + d_i)_+ - (e_i - bQ - b - d_i)_+}{b + 2d_i} \\ &\leq \frac{2(b + d_i)}{b + 2d_i} = 1 + \frac{b}{b + 2d_i} < 2, \end{aligned}$$

and so $\lfloor q_i^+(Q) \rfloor \leq \lceil q_i^-(Q) \rceil + 1$, with equality only if

$$\frac{(e_i - bQ + b + d_i)_+}{b + 2d_i} = z + f$$

for some $z \in \mathbb{Z}, z > 0$ and $0 \leq f \leq b/(b+2d_i)$, and $\lfloor q_i^+(Q) \rfloor = z$, $\lceil q_i^-(Q) \rceil = z-1$. But then

$$q_i^+(Q+j) = (z + f - \frac{2b}{b+2d_i})_+ < z,$$

and so again the claim is proved.

For the last part, we consider first the total production. For any n let k be a positive integer, let $a = (k+1)n+1$, $b=1$, c a vector of ones, and d $kn-1/2$ times c. Then, for each i,

$$\frac{(e_i - bQ \pm (b+d_i))_+}{b+2d_i} = \frac{((k+1)n - Q \pm (kn+1/2))_+}{2kn},$$

and for $Q = n$, these values are 0 and $1+(4kn)^{-1}$. Hence there is an integer Cournot equilibrium where each producer chooses $q_i = 1$. However, in the real Cournot equilibrium, the total production Q^* satisfies

$$Q^* = n\frac{(k+1)n - Q^*)_+}{2kn},$$

which gives $Q^* = (k+1)n/(2k+1) = n/2 + n/(4k+2)$. Choosing $k > n/(4\epsilon)$ gives $Q^* < n/2 + \epsilon$, more than $n/2 - \epsilon$ from the integer Cournot equilibrium with total production n. If we change a to $kn+1$, the integer Cournot equilibrium has total production 0, whereas this is $kn/(2k+1) = n/2 - n/(4k+2)$ in the unique real Cournot equilibrium.

Lastly, we construct an example where the first producer chooses a level in an integer Cournot equilibrium which is far from her choice in the real equilibrium. We suppose $n = \bar{n}+1 = 4m+1$ for some integer m. We choose $a = (k+1/4)\bar{n}+1$ and $b=1$. For the first producer, we choose $c_1 = (k-1/4)\bar{n}+1$ and $d_1 = 0$, while the remaining $\bar{n}$ producers have $c_i = 1$ and $d_i = k\bar{n} - 1/2$.

First, $q = (\bar{n}/4, 0, \ldots, 0)$ is an integer Cournot equilibrium. Indeed, for $i=1$ we have $(e_1 - bQ \pm (b+d_1))_+/(b+2d_1) = (\bar{n}/2 - \bar{n}/4 \pm 1)_+ = \bar{n}/4 \pm 1$, and q_1 lies in this range. For $i > 1$, $(e_i - bQ \pm (b+d_i))_+/(b+2d_i) = ((k+1/4)\bar{n} - \bar{n}/4 \pm (k\bar{n}+1/2))_+/(2k\bar{n})$ and these values are 0 and $1+1/(4k\bar{n})$. Hence an equilibrium choice has to be 0 or 1, and $q_i = 0$ is possible.

Now let Q^* be the total production in the real equilibrium, and suppose all q_i^*'s are positive. Then for $i=1$ we have $q_1^* = (e_1 - bQ^*)/(b+2d_1) = \bar{n}/2 - Q^*$, while for $i > 1$ we obtain $q_i^* = (e_i - bQ^*)/(b+2d_i) = ((k+1/4)\bar{n} - Q^*)/(2k\bar{n})$, and the sum of all these equals Q^* iff

$$\left(\frac{\bar{n}}{2} - Q^*\right) + \frac{(k+1/4)\bar{n} - Q^*}{2k} = Q^*, \text{ or } Q^* = \frac{2k+1/4}{4k+1}\bar{n},$$

and then all q_i^* are indeed positive, with $q_1^* = (1/(16k+4))\bar{n}$, and we see that the distance between q_1 and q_1^* is greater than $\bar{n}/4 - \epsilon$ for sufficiently large k.

This completes the proof of the theorem. $\square$

Three comments on the result are in order:

Firstly, if one is concerned that the multiplicity of equilibria is only due to the fact that all producers in the first example in the proof are identical, we note that the first set of equilibria arises even if all the c_i's are distinct numbers in the interval $[1, 2)$. Secondly, there is a gap between our bound $(n/2+1)$ on the distance of an individual producer's choice in an integer Cournot equilibrium from her choice in the real equilibrium, and our example showing that the distance can be about $n/4$. We suspect that with a more sophisticated example we could show that distance close to $n/2$ is possible.

Thirdly, and most importantly, it might be thought that the large distance from the real equilibrium is due to the coarse granularity of the integers around 0, since that is a feature in our examples. However, by adjusting the values of a and the c_i's, we can translate each e_i by a positive integer multiple of $b + 2d_i$. This translates all equilibria, integer and real, by the same vector of these integers. Hence we can once again find discrepancies of $n/2$ in the total production, or $n/4$ in the choice of an individual producer, even if these levels are large. This shows the possibility of a large relative gap between an individual producer's choice in an integer as opposed to the real Cournot equilibrium as long as the number of producers is comparable to this individual production level.

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