OR 635: Interior Point Methods for Math Progg. Fall 2003.

Homework Set 4. Due: Friday December 5.

1. Suppose Δx is calculated as in Ye's primal-dual potential reduction algorithm using Φ_ρ and $\|\Delta x\|_x < 1/4$. Set $s_+ := -\lambda F'(x) - \lambda F''(x)\Delta x$ with $\lambda := \langle s, x\rangle/\rho$. Let $(x(\lambda), y(\lambda), s(\lambda))$ be the point on the central path corresponding to λ, so $x(\lambda)$ uniquely minimizes $\psi_\lambda(\hat{x}) := \frac{1}{\lambda}\langle c, \hat{x}\rangle + F(\hat{x})$.

a) Let $\epsilon := \|x(\lambda) - x\|_x$ and, assuming ϵ is positive, $h := \frac{1}{\epsilon}(x(\lambda) - x)$, so that $x(\lambda) = x + \epsilon h$. Define $\theta_\lambda(\tau) := \psi_\lambda(x + \tau h)$. Show that θ_λ is self-concordant on an interval of the real line containing 0 in its interior, that $\theta''_\lambda(0) = 1$, and that $|\theta'_\lambda(0)| < 1/4$. Deduce that θ_λ is minimized at some τ with $|\tau| \leq 1/2$.

b) Show that $\|x(\lambda) - x\|_x \leq 1/2$ and

$$\begin{aligned} |\langle s, x(\lambda)\rangle - \langle s, x\rangle| &= |\langle s_+, x(\lambda) - x\rangle| \\ &= |\langle s_+ + \lambda F'(x) - \lambda F'(x), x(\lambda) - x\rangle| \\ &\leq \lambda \cdot (\sqrt{\nu} + \frac{1}{4}) \cdot \frac{1}{2}. \end{aligned}$$

2. Now compute $(\widehat{\Delta x}, \widehat{\Delta y}, \widehat{\Delta s})$ by

$$\begin{array}{rcrcl} A^*\widehat{\Delta y} & + & \widehat{\Delta s} & = & 0, \\ A\widehat{\Delta x} & & & = & 0, \\ \widehat{\Delta x} & + & F''_*(s)\widehat{\Delta s} & = & -\frac{x}{\lambda} - F'_*(s). \end{array} \tag{1}$$

Argue that, if $\|\widehat{\Delta s}\|^*_s < 1/4$, then $|\langle s(\lambda), x\rangle - \langle s, x\rangle| \leq \lambda \cdot (\sqrt{\nu} + \frac{1}{4}) \cdot \frac{1}{2}$.

3. a) Show that

$$\langle s(\lambda), x(\lambda)\rangle = -\langle s, x\rangle + \langle s(\lambda), x\rangle + \langle s, x(\lambda)\rangle.$$

b) (Gonzaga and T.) Suppose that $\rho \geq \nu + 2\sqrt{\nu}$. Prove that either $\|\Delta x\|_x \geq 1/4$ or $\|\widehat{\Delta s}\|^*_s \geq 1/4$.

4. (Tunçel) Given that $F \in LHSCB(K, \nu)$, F is self-scaled as long as
(i) For all $x, w \in \text{int } K$, $F''(w)x \in \text{int } K^*$, and
(ii) For all such x, w, $F_*(F''(w)x) = F(x) - 2F(w) - \nu$.

a) Show that if F is self-scaled,
(iii) For all $x, v, w \in \text{int } K$, if $F''(w)x = -F'(v)$, then $F(w) = (F(v) + F(x))/2$.

b) Show that (iii) can replace (ii) in the definition of self-scaledness.

c) The conclusion of (iii) is symmetric between x and v. In fact, so is the hypothesis: show that if F is self-scaled and $F''(w)x = -F'(v)$, then $F''(w)v = -F'(x)$.