OR 635: Interior Point Methods for Math Progg. Fall 2003.

Comments on Homework Set 3.

You generally did OK on questions 1, 2, and 3. You need to be a little careful in writing your arguments – for example, it is good to point out that $1/4$ lies in int C because it lies within 1 of 0 in 0's local norm.

Also, you were generally fine on Q5 (the directions when the current iterate is on the central path). For this it is much cleaner to argue using the linear systems solved by the two directions rather than the directions themselves, which involve more complicated formulae. For instance, $(\Delta x_1, \Delta y_1, \Delta s_1)$ is the unique solution to

$$A^* \Delta y + \Delta s = 0, A\Delta x = 0, \text{ and } \mu\gamma F''(x)\Delta x + \Delta s = -(1-\gamma)s,$$

while a little manipulation shows that $(\Delta x_2, \Delta y_2, \Delta s_2)$ is the unique solution to

$$A^* \Delta y + \Delta s = 0, A\Delta x = 0, \text{ and } \mu F''(x)\Delta x + \gamma\Delta s = -(1-\gamma)s.$$

From this it is easy to conclude that $\Delta x_2 = \gamma\Delta x_1$ and $(\Delta y_2, \Delta s_2) = (1/\gamma)(\Delta y_1, \Delta s_1)$.

Here is a solution to Q4:

(i) (Here $[\alpha, \infty]$ should have been $[\alpha, \infty)$.) If $\phi'(\xi) \geq 0$ for some $\xi \in \text{int } C$, then by Q2 we would have

$$\{\xi + \tau \in C : \tau \geq 0\} \subseteq \{\xi + \tau \in C : \tau\phi'(\xi) \geq 0\} \subseteq \{\xi + \tau : \phi''(\xi)\tau^2 \leq (6\nu + 1/4)^2\},$$

a bounded set, contrary to hypothesis. Hence $\phi'(\xi) < 0$ for all such ξ.

Now we use Corollary 2 of the theorem on the Taylor approximation of F' (see the notes for 11/10).There is no ξ with $\phi'(\xi) = 0$, so ϕ can be reduced by $1/48$ from any point by taking a damped Newton step. This implies that ϕ is unbounded below, and hence tends to $-\infty$ as ξ tends to ∞. (Van Anh also had a pretty alternative argument.)

(ii) (We need to assume $\bar{x}$ is nonzero, of course.) We define $\phi(\tau) := F(\hat{x} + \tau\bar{x})$. Since $\hat{x} + \tau\bar{x} \in \text{int } K$ for all nonnegative τ, the domain of ϕ is of the form $[\alpha, \infty)$. (If the domain is $(-\infty, \infty)$, then $\hat{x} + \tau\bar{x} \in \text{int } K$ for all real τ, and then by dividing by τ or $-\tau$ and taking limits, we find $\bar{x} \in K \cap (-K)$, which contradicts K being pointed.) Now the conclusion follows from part (i).

(iii) (This is Ramu's nice solution.) It is *not* enough to show that $F_*(s_\infty) = +\infty$, since we cannot assume that F_* has the barrier property or is continuous. We need to prove it directly. So choose $\bar{x} \in K \setminus \{0\}$ with $\langle s_\infty, \bar{x} \rangle = 0$, and choose any $\hat{x} \in \text{int } K$. Then

$$\begin{aligned} F_*(s_k) &= \sup\{-\langle s_k, x\rangle - F(x)\} \\ &\geq -\langle s_k, \hat{x} + \tau\bar{x}\rangle - F(\hat{x} + \tau\bar{x}) \\ &= -\langle s_k, \hat{x}\rangle - \tau\langle s_k, \bar{x}\rangle - F(\hat{x} + \tau\bar{x}), \end{aligned}$$

for any positive τ.

Now let $\epsilon_k := \langle s_k, \bar{x}\rangle > 0$, since $\bar{x} \in K \setminus \{0\}$ and $s_k \in \text{int } K^*$. Then $\epsilon_k \to \langle s_\infty, \bar{x}\rangle = 0$. Choose $\tau = 1/\epsilon_k$ above to get

$$F_*(s_k) \geq [-\langle s_k, \hat{x}\rangle] + [-1] + [-F(\hat{x} + (1/\epsilon_k)\bar{x})].$$

As k tends to infinity, the first term converges to the finite value $-\langle s_\infty, \hat{x}\rangle$ while the last tends to $+\infty$ by part (ii), establishing the desired result.