OR 635: Interior Point Methods for Math Programming. Fall 2003.

Homework Set 2. Due: Friday October 17.

1. Suppose that x and (y, s) are on the central path for the LP problem (P). By scaling the data A, b, and c, we can assume that $x = s = e$ and $\mu = 1$. Let v_p denote the projection of $v \in R^n$ onto the null space of$A$.

a) Show that, if (d_x, d_y, d_s) is the Newton step corresponding to $\mu^+ < 1$, then the directions of d_x and d_s are independent of μ^+ and can be written without reference to A, b or c (except for using projections).

b) Let $x(\alpha) := e - \alpha e_p$, $s(\alpha) := e - \alpha(e - e_p)$. Show that $x(\alpha) > 0$ and $s(\alpha) > 0$ for $\alpha < 1/\sqrt{n}$. By considering the vectors $u := (\frac{1+\sqrt{n}}{2}, \frac{1}{2}, ..., \frac{1}{2})^T$, $v := (\frac{1-\sqrt{n}}{2}, \frac{1}{2}, ..., \frac{1}{2})^T$, demonstrate that this bound is close to being tight.

c) Find an expression (not involving A, b or c) for the improvement in the primal objective function in going from $x(0)$ to $x(\alpha)$, and similarly for the dual. Show that the sum of these improvements is αn, but for given α either one could be arbitrarily small.

d) Suppose $\mathrm{Diag}(e_p).(e - e_p) = 0$. What are optimal solutions to (P) and its dual (s variables only)?

2. Let $\Phi_\rho(x, s)$ be the primal-dual potential function $\rho \ln(x^T s) - \ln(x) - \ln(s)$, with $\rho \geq \sqrt{n}$. At the point $x = s = e$ as in the previous question, show that either $P_A \nabla_x \Phi_\rho(e, e)$ or $(I - P_A)\nabla_s \Phi_\rho(e, e)$ has Euclidean length at least $\frac{1}{\sqrt{2}}$. (This suggests that a fixed decrease in the potential function can be obtained by taking either a primal or a dual scaled steepest descent step. Indeed this is so, even if (x, s) is not centered (Gonzaga and Todd).)

3. Let K and K^* be dual convex cones in E and E^* respectively.

a) Suppose $s \in$ int K^*. Show that, for all nonzero $x \in K$, $\langle s, x\rangle > 0$.

b) With s as in (a), show that, for all $\alpha > 0$, $\{x \in K : \langle s, x\rangle \leq \alpha\}$ is bounded.

c) Show that, if K^* is solid, K is pointed, and hence that if K is solid, K^* is pointed.

d) Show that, if K is pointed, K^* is solid, and hence that if K is pointed, K^* is solid. (As in class, you can use the fact that if the convex cone K^* is not solid, its affine (and linear) hull is a proper subspace of E^*.)

4. For the conic programming problems (P) and (D), suppose that $\mathcal{F}^0(P)$ and $\mathcal{F}^0(D)$ are nonempty. Prove that both (P) and (D) have nonempty bounded sets of optimal solutions (under the blanket assumption that A is surjective).