OR635 Final Exam Fall ’03
Interior-Point Methods 24 hour take-home

You should bring your solution back to Donna in ETC 204 within 24 hours of picking it up. Do not discuss this with anyone until after noon on December 12. You may use any results from class notes or homework, or any clearly-stated standard result from linear algebra or analysis. If you have any questions or need hints, you can send me e-mail (miketodd@cs.cornell.edu) or call me (255-9135, office, 257-3344, 0800-2200, home).

Conciseness counts. Please do not redo all the proofs from class in question 2; it is enough to show what is new and what it implies.

OVER

1. For LP, we considered the path $x(\mu)$ of solutions to the barrier problem:

$$(BP)_\mu \qquad \begin{array}{rcl} \min c^\top x & - & \mu \sum_j \ln(x_j) \\ Ax & = & b \\ x & > & 0. \end{array}$$

This question is concerned with solutions to the weighted barrier problem

$$(WBP)_\mu \qquad \begin{array}{rcl} \min c^\top x & - & \mu \sum_j w_j \ln(x_j) \\ Ax & = & b \\ x & > & 0, \end{array}$$

where $w_j \geq 0$, $j = 1, \ldots, n$.

Let $x \in \mathcal{F}^0(P)$.

a) Suppose x^* and (y^*, s^*) are optimal solutions to (P) and (D) such that the columns of A in $\{a_j : s_j^* = 0\}$ are linearly independent. Show that, if $w_j = x_j s_j^*$ for all j, $(WBP)_\mu$ has the unique solution

$$x(\mu, w) := x^* + \mu(x - x^*)$$

for $\mu > 0$. Hence the path is now a straight line to the optimal solution.

b) Of course, s^* is unknown. Suppose some $(y, s) \in \mathcal{F}^0(D)$ is known, and let $w_j = x_j s_j$ for all j. Show that x then solves $(WBP)_\mu$ for $\mu = 1$. Prove that the negative of the tangent to the path of solutions of $(WBP)_\mu$ for $\mu > 0$, at the point x, points in the direction d_x that is the primal-dual affine-scaling direction for x using the symmetric scaling $W = X^{1/2} S^{-1/2}$. (For example, you can use the implicit function theorem.) (This gives a primal justification for this scaling.)

2. Consider Ye's primal-dual potential-reduction method. In the notes, we took $\rho = \nu + \sqrt{\nu}$ in $\Phi_\rho(x, s) := \rho \ln\langle s, x\rangle + F(x) + F_*(s)$; now we consider other choices. Recall that, for $(x, s) \in \mathcal{F}^0(P) \times \mathcal{F}^0(D)$, if $\|\Delta x\|_x < 1/4$, we set $s_+ := -\lambda F'(x) - \lambda F''(x)\Delta x$, where $\lambda := \langle s, x\rangle/\rho$.

a) If $\rho = \nu + \kappa\sqrt{\nu}$, with κ an absolute constant at least 1, show that

$$\Phi_\rho(x, s_+) \le \Phi_\rho(x, s) - \frac{1}{6}.$$

b) If $\rho = \kappa\nu$, with κ an absolute constant at least 2, show that

$$\Phi_\rho(x, s_+) \le \Phi_\rho(x, s) - \frac{\nu}{3}.$$

(You can use $\ln(2) > 2/3$.)

c) Does the result of (b) lead to an $O(\ln(1/\epsilon))$-iteration algorithm? Why or why not? What are the iteration complexities of Ye's algorithm with ρ as in (a) and (b)?

3. We defined (equivalently) the primal central path as the set of minimizers of the penalized function $\langle c, x\rangle + \mu F(x)$ over $\mathcal{F}^0(P)$ as $\mu > 0$ varies. Here we consider whether it can be defined as the set of minimizers of the primal potential function $\phi_\rho(x; \zeta) := \rho \ln(\langle c, x\rangle - \zeta) + F(x)$ over $\mathcal{F}^0(P)$ as ζ varies. Assume throughout that $\mathcal{F}^0(P)$ and $\mathcal{F}^0(D)$ are nonempty.

a) Suppose $\rho > \nu$. Let $\zeta < \zeta_* := v(P) = v(D)$. Show that any minimizer of $\phi_\rho(x; \zeta)$ over $\mathcal{F}^0(P)$ lies on the primal central path.

b) Suppose again $\rho > \nu$. Show that every point on the primal central path is the minimizer (not just a stationary point!) of $\phi_\rho(x; \zeta)$ over $\mathcal{F}^0(P)$ for some $\zeta < \zeta_*$. (Hint: the hard part is proving that minimizers exist.)

c) State a result connecting the primal-dual central path with minimizers of the primal-dual potential function Φ_ρ.

OVER

4. Here are two fun problems:

a) The “post office problem” is to find the dimensions of the largest box whose girth plus length is at most 72 inches, or

$$\max\{x_1x_2x_3 : x_1 + 2x_2 + 2x_3 \leq 72, x_1, x_2, x_3 \geq 0\}.$$

Argue that the (first) inequality can be replaced by an equation. Next show that the solution is the analytic center of a certain polyhedron. Hence solve the problem. (This may help any of you sending off gift packages.)

b) The problem of finding the rectangle of minimum perimeter whose area is at least 1 can be written as

$$\min\{2x_1 + 2x_2 : x_1x_2 \geq 1, x_1, x_2 \geq 0\}.$$

Reformulate this as a second-order cone programming problem, and hence solve it. (This might help you wrap up a gift in the smallest amount of paper, if you happen to live in Flatland.)