LMI approximations for cones of positive semidefinite forms

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Abstract

An interesting recent trend in optimization is the application of semidefinite programming techniques to new classes of optimization problems. In particular, this trend has been successful in showing that under suitable circumstances, polynomial optimization problems can be approximated via a sequence of semidefinite programs. Similar ideas apply to conic optimization over the cone of copositive matrices, and to certain optimization problems involving random variables with some known moment information.

In this paper we address the approximability of cones of *positive semidefinite forms* (homogeneous polynomials). This perspective allows us to extract some key common ideas underlying the approximations mentioned above. At the same time, our approach suggests natural new approximation schemes. In particular, we show that for several interesting cones of positive semidefinite forms it is possible to construct polyhedral approximations, which opens the possibility of using linear programming technology in optimization problems over these cones.

1 Introduction

An interesting recent trend in optimization is the use of semidefinite programming techniques for solving, or approximating new classes of optimization problems. In particular, Lasserre [13] proposed a solution approach for polynomial optimization problems. Independently, Parrilo [19, 20] developed sufficient criteria based on *sum-of-squares* to test whether a matrix is copositive. A key ingredient underlying both of these approaches, rooted on previous work by

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Shor [30, 31] and Nesterov [18], is the idea of approximating the cone of positive semidefinite polynomials (non-negative polynomials) via the cone of positive semidefinite matrices.

In this paper we develop approximations for the cone $P_{n,m}(D)$ of positive semidefinite forms (homogeneous polynomials) of degree m over a conic domain $D \subseteq \mathbb{R}^n$. Some well-known cones of psd forms are the cone $P_{n,2}(\mathbb{R}^n)$ of positive semidefinite matrices, and the cone $P_{n,2}(\mathbb{R}^n_+)$ of copositive matrices. Also, via homogenization, cones of non-negative polynomials correspond to $P_{n,m}(D)$ for suitable choices of n, m, and D (see Section 2.2). One of the most interesting features of addressing the approximability of cones of positive semidefinite forms is that it reveals some simple common ideas at the core of the constructions independently proposed by Lasserre [13, 14] and Parrilo [19, 20], among others. It also suggests some natural approximations schemes that have been somewhat overlooked so far. In particular, we give an approximation scheme for $P_{n,2m}(\mathbb{R}^n)$ and some polyhedral approximation schemes for $P_{n,m}(D)$ for pointed polyhedral conic domains D.

A wide variety of optimization problems can be cast as conic programs (cf. [25]) over cones of positive semidefinite forms. For example, unconstrained polynomial optimization can be cast as a conic program over $P_{n,m}(\mathbb{R}^n)$ (see Example 1). Some classes of polynomially constrained polynomial optimization problems can also be cast as a conic program over $P_{n,m}(D)$ for a suitable domain D (see Example 2). The problem of computing the stability number of a graph can be cast as an optimization problem over the cone of copositive matrices (see [3]). Furthermore, conic duality shows that the dual cone $P_{n,m}(D)^*$ corresponds to the cone of m-order moments of finite measures supported in D. Thus, a large subclass of generalized Tchebycheff inequalities (cf. [11]) can be studied as optimization problems over the cone $P_{n,m}(D)^*$ (see [35]).

Given that $P_{n,m}(D)$ unifies such diverse optimization problems, is not surprising that simply checking membership in $P_{n,m}(D)$ is in general an intractable problem (see [17]). The main goal of this paper is to show that in spite of this difficulty, in many important cases it is possible to construct approximation schemes for the cone $P_{n,m}(D)$ that readily yield conceptual algorithms for finding arbitrarily approximate solutions to optimization problems over this cone. More explicitly, for a large class of conic domains D, we construct an increasing sequence of cones K^r, $r = 0, 1, \ldots$ with the following three properties: First, K^r, $r = 0, 1, \ldots$ gives an increasing sequence of *inner approximations* for $P_{n,m}(D)$. Second, in the limit this sequence contains the interior of $P_{n,m}(D)$. Third, K^r, $r = 0, 1, \ldots$ can be described via the cone of positive semidefinite matrices or the positive orthant. As a consequence, under suitable hypotheses, it is possible to compute an arbitrarily close approximation for a conic program over the cone $P_{n,m}(D)$ by solving the conic programs obtained when $P_{n,m}(D)$ is replaced by K^r (cf. Theorem 3 and Example 1 below).

The fundamental tools used to construct these inner approximations for $P_{n,m}(D)$ are representation theorems for positive definite forms and polynomials from real algebraic geometry. Therefore, our work builds up on the recent idea of connecting results of algebraic geometry with optimization (see, e.g., [13, 14, 15, 16, 19, 20]).

Besides the work by Lasserre and Parrilo, the idea of approximating the cone of positive semidefinite polynomials is implicit in the work by Bertsimas and Popescu [1], de Klerk [3], Popescu [22], and Kojima et al. [12]. Our treatment allow us to connect and extend several key ideas developed in these papers. In particular, we propose a natural generalization of the (sufficient) criterion for copositivity proposed by Parrilo in [19], and show that similar results hold for more general classes of cones of positive semidefinite forms. Moreover, we show that in many important cases, besides *semidefinite* approximations; that is, approximations via the cone of positive semidefinite matrices, one can obtain *polyhedral* approximations; that is approximations via the positive orthant, for $P_{n,m}(D)$. This important fact has received little attention in previous work (one of the few exceptions is [3]). Although polyhedral approximations for $P_{n,m}(D)$ are in general weaker (inclusion-wise) than semidefinite approximations; they open possibilities for use of the highly developed linear programming technology. Given the limitations of current semidefinite programming solvers to handle large-scale problems, the availability of polyhedral approximations can potentially yield substantial enhancements in the solution techniques for problems involving cones of positive semidefinite forms.

The rest of the paper is organized as follows. In Section 2 we introduce some key definitions and present Theorem 2, which summarizes our main results. Also, we briefly discuss the correspondence between cones of positive semidefinite polynomials and positive semidefinite forms. In Sections 3 and 4 we construct inner approximations for the cones $P_{n,m}(\mathbb{R}^n)$ and $P_{n,m}(\mathbb{R}^n_+)$ respectively. In the latter case, which generalizes the cone of copositive matrices, along with a semidefinite approximation, we construct a polyhedral approximation for this cone. In Section 5 we generalize the construction and key results from Section 4 to the cone $P_{n,m}(D)$ when D is a pointed polyhedral cone. Section 6 discusses similar results for the more general cone of positive semidefinite forms over pointed semialgebraic cones. Throughout our presentation we emphasize the two key ideas underlying these approximations; namely: *approximate* $P_{n,m}(D)$ with simply described sets, and *embed* $P_{n,m}(D)$ in a higher dimensional space. The last section of the paper addresses the dual counterpart of the cone $P_{n,m}(D)$; namely, *the cone of moments* $M_{n,m}(D)$. In particular, we use conic duality combined with the inner approximations derived in Sections 3 through 6 to obtain *outer approximations* for the cone $M_{n,m}(D)$ for several conic domains D.

2 Preliminaries and main result

2.1 Monomials, forms, cones of positive semidefinite forms

We begin by recalling some standard multinomial notation and terminology. Given $\alpha := (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ and a vector of variables $x := (x_1, \ldots, x_n)$ the expression x^α denotes the monomial

$$x_1^{a_1} \cdots x_n^{\alpha_n}.$$

We also write $|\alpha|$ for $\alpha_1 + \cdots + \alpha_n$.

Let $H_{n,m}$ denote the set of forms (homogeneous polynomials) of degree m in n variables with real coefficients. A form $\theta(x)$ in $H_{n,m}$ can be written as

$$\theta(x) = \sum_{|\alpha|=m} \theta_\alpha x^\alpha.$$

We shall identify the form $\theta(x)$ with the vector of its coefficients $\theta := (\theta_\alpha)_{|\alpha|=m}$. Formally speaking, θ denotes the *vector* $(\theta_\alpha)_{|\alpha|=m}$, and for a given $x \in \mathbb{R}^n$, $\theta(x)$ denotes the *value* $\sum_{|\alpha|=m} \theta_\alpha x^\alpha$, i.e., the value of the form θ evaluated at x. Via this identification, the set $H_{n,m}$ can in turn be identified with the Euclidean space $\mathbb{R}^{n_m}$, where

$$n_m := |\{\alpha \in \mathbb{N}^n : |\alpha| = m\}| = \binom{n+m-1}{n-1}.$$

We will make extensive use of this identification. In particular, endow $H_{n,m}$ with the dot-inner product. In other words, define the inner product of $\theta, \phi \in H_{n,m}$ as

$$\langle \theta, \phi \rangle := \sum_{|\alpha|=m} \theta_\alpha \phi_\alpha.$$

We shall also frequently use the vector-valued function $\sigma^m : \mathbb{R}^n \to \mathbb{R}^{n_m}$ defined by

$$x \mapsto (x^\alpha)_{|\alpha|=m}.$$

Notice that by construction, the following identity holds for all $\theta \in H_{n,m}$ and $x \in \mathbb{R}^n$.

$$\theta(x) = \langle \theta, \sigma^m(x) \rangle.$$

For the special case $m = 2$, the space of 2-forms $H_{n,2} = \mathbb{R}^{n_2}$ can also be identified with the space $\mathcal{S}^n$ of $n \times n$ symmetric matrices. The identification is via the one-to-one correspondence between symmetric matrices and quadratic forms $Q \in \mathcal{S}^n \mapsto q \in H_{n,2}$, where $q(x) := x^{\mathrm{T}} Q x$.

Definition 1 Given a cone $D \subseteq \mathbb{R}^n$, let $P_{n,m}(D)$ be the cone of m-degree forms that are *positive semidefinite* in D (psd in D), i.e.,

$$P_{n,m}(D) := \{\theta \in H_{n,m} : \theta(x) \geq 0 \text{ for all } x \in D\}.$$

When $m = 2$, $D = \mathbb{R}^n$, and identifying $H_{n,2}$ with $\mathcal{S}^n$, the cone $P_{n,2}(\mathbb{R}^n)$ corresponds precisely to the cone of positive semidefinite matrices, usually denoted $\mathcal{S}^n_+$. We shall write $A \succeq 0$ for $A \in \mathcal{S}^n_+$ following the usual notation in the semidefinite programming literature (see e.g., [33]).

If $\theta \in P_{n,m}(D)$ satisfies $\theta(x) > 0$ for all $x \in D$, $x \neq 0$, then p is said to be *positive definite* in D (pd in D). Throughout our presentation we will frequently rely on the following straightforward characterization of the interior of $P_{n,m}(D)$.

Observation 1 *Assume $D \subseteq \mathbb{R}^n$ is a closed cone. Then θ is pd in D if and only if $\theta \in \mathrm{int}(P_{n,m}(D))$.*

Let us introduce one more piece of notation. Let $\Sigma_{n,2m}$ denote the cone of forms in $H_{n,2m}$ that are sum of squares (sos), that is,

$$\Sigma_{n,2m} := \mathrm{conv}\{\phi(x)^2 : \phi \in H_{n,m}\}.$$

(Here $\mathrm{conv}(S)$ denotes the convex hull of the set S.)

Notice that $\Sigma_{n,2m} \subseteq P_{n,2m}(\mathbb{R}^n)$ for all m, n. Except for some special cases, this inclusion is proper. This is a classical result due to Hilbert [8].

Theorem 1 (Hilbert) $\Sigma_{n,2m} = P_{n,2m}(\mathbb{R}^n)$ *if and only if* $n \leq 2$, $m \leq 1$, *or* $(n, m) = (3, 2)$.

The study of the relationships between psd forms and sos has a long history. The search for such kinds of connections is closely tied with Hilbert's 17th problem [9] and with advances in real algebra over the last century ([23, 27]). Our work relies on some of these developments. For a detailed account of the rich history of this subject, we refer the reader to the excellent references [23, 26, 27].

Theorem 2 below summarizes the key results of this paper: For several important classes of conic domains D, the cone $P_{n,m}(D)$ can be innerly approximated by a sequence of cones definable in terms of linear matrix inequalities (LMI).

Theorem 2 *Suppose $D = \mathbb{R}^n$, $D = \mathbb{R}^n_+$ or D is a pointed semialgebraic cone. Then we can construct a sequence of cones $K^r, r = 0, 1, \ldots$ such that*

(i) $K^r \subseteq K^{r+1} \subseteq P_{n,m}(D)$, $r = 0, 1, \ldots$

(ii) $\mathrm{int}(P_{n,m}(D)) \subseteq \bigcup_{r=0}^{\infty} K^r$

(iii) *Each K^r has an LMI description. In other words,*

$$K^r = \{\theta \in H_{n,m} : \exists \Phi \succeq 0 \quad \text{s.t.} \;\; \mathcal{L}\theta = \mathcal{T}\Phi\} \tag{1}$$

for some suitable linear maps $\mathcal{L}, \mathcal{T}$.

(iv) *If D is a polyhedral cone, then we can also construct a polyhedral sequence of cones approximating $P_{n,m}(D)$ as above (i.e. we set $\Phi \geq 0$ in (1)).*

Proof. See Propositions 1, 2, and 9 in Sections 3, 4, and 6 respectively. □

Throughout the sequel, we shall use $K^r \uparrow P_{n,m}(D)$ as a shorthand for conditions (i) and (ii) in Theorem 2. Also, whenever we say that $\{K^r, r = 0, 1, \ldots\}$ is a *sequence of inner approximations* for $P_{n,m}(D)$ it will be implicitly assumed that $\{K^r, r = 0, 1, \ldots\}$ satisfies conditions (i), (ii), and (iii) in Theorem 2.

Theorem 2 in combination with Theorem 3 below readily yield SDP-based numerical schemes for computing arbitrarily close approximate solutions to primal-dual pairs of conic programs of the form

$$(\mathbb{P}) \quad \begin{array}{rll} z_{\mathbb{P}} = & \inf & \langle c, \theta \rangle \\ & \text{s.t.} & A\theta = b \\ & & \theta \in P_{n,m}(D) \end{array} \qquad (\mathbb{D}) \quad \begin{array}{rll} z_{\mathbb{D}} = & \sup & \langle b, v \rangle \\ & \text{s.t.} & c - A^* v \in P_{n,m}(D)^*, \end{array}$$

for the classes of conic domains D in Theorem 2. Here A^* is the adjoint of A and $P_{n,m}(D)^*$ is the dual cone of $P_{n,m}(D)$ (see Section 7). This general format underlies several of the ideas and results in [13, 14, 19].

Consider the primal-dual pair of conic programs obtained when $P_{n,m}(D)$ is replaced by K^r in $(\mathbb{P})$ and $(\mathbb{D})$:

$$(\mathbb{P}_r) \quad \begin{array}{rll} z_{\mathbb{P}_r} = & \inf & \langle c, \theta \rangle \\ & \text{s.t.} & A\theta = b \\ & & \theta \in K^r \end{array} \qquad (\mathbb{D}_r) \quad \begin{array}{rll} z_{\mathbb{D}_r} = & \sup & \langle b, v \rangle \\ & \text{s.t.} & c - A^* v \in (K^r)^*, \end{array}$$

Theorem 3 below formalizes the intuitively natural fact that $(\mathbb{P})$ and $(\mathbb{D})$ are approximated when the cone $P_{n,m}(D)$ is suitably approximated by a sequence of cones. This result can be seen as a strengthening of a classical strong conic duality theorem in convex analysis (cf. [25, 28]).

Theorem 3 *Assume* $(\mathbb{D})$ *is feasible,* A *is surjective, and* $(\mathbb{P})$ *is strictly feasible (i.e., there exists* $\theta \in \operatorname{int}(P_{n,m}(D))$ *such that* $A\theta = b$*). Let* $K^r \uparrow P_{n,m}(D)$ *and* $(\mathbb{P})$, $(\mathbb{P}_r)$, $(\mathbb{D})$, $(\mathbb{D}_r)$ *be as above. Then*

(i) $(\mathbb{D}_r)$ *is feasible and* $z_{\mathbb{D}_r} \geq z_{\mathbb{D}_{r+1}} \geq z_{\mathbb{D}}$ *for* $r = 0, 1, \ldots$.

(ii) *For* r *sufficiently large,* $z_{\mathbb{P}_r} = z_{\mathbb{D}_r}$ *and* $(\mathbb{D}_r)$ *has an optimal solution* v^r.

(iii) $\lim_{r\to\infty} z_{\mathbb{D}_r} = z_{\mathbb{D}} = z_{\mathbb{P}}$*, the set* $\{v^r : r = 0, 1, \ldots\}$ *is bounded, and every limit point of* $\{v^r : r = 0, 1, \ldots\}$ *is an optimal solution of* $(\mathbb{D})$.

Proof. This is a dual version of [35, Thm. 2] and can be proven by a similar argument. □

2.2 Positive semidefinite polynomials over sets

We next point out a one-to-one correspondence between cones of positive semidefinite polynomials over arbitrary sets and cones of positive semidefinite forms over conic domains. This correspondence readily follows from a natural homogenization. As an immediate consequence of this one-to-one correspondence it follows that the class of cones of positive semidefinite polynomials can be seen as a subclass of cones of positive semidefinite forms. Consequently, our presentation focuses on the latter class of cones.

Let $\mathcal{H}_{n,m}$ denote the set of polynomials of degree at most m in n variables. A polynomial $g(x)$ in $\mathcal{H}_{n,m}$ can be written as

$$g(x) = \sum_{|\alpha| \le m} g_\alpha x^\alpha.$$

Given a set $\mathcal{D} \subseteq \mathbb{R}^n$, let $\mathcal{P}_{n,m}(\mathcal{D})$ denote the cone of polynomials in $\mathcal{H}_{n,m}$ that are positive semidefinite in $\mathcal{D}$, i.e.,

$$\mathcal{P}_{n,m}(\mathcal{D}) := \{g \in \mathcal{H}_{n,m} : g(x) \ge 0 \text{ for all } x \in \mathcal{D}\}.$$

The spaces $\mathcal{H}_{n,m}$ and $H_{n+1,m}$ are isomorphic via the homogenization

$$g(x) = \sum_{|\alpha| \le m} g_\alpha x^\alpha \mapsto \breve{g}(x, x_{n+1}) = \sum_{|\alpha| \le m} g_\alpha x^\alpha x_{n+1}^{m-|\alpha|}.$$

In other words, the isomorphism is $g \mapsto \breve{g}$, where

$$\breve{g}_{(\alpha,\alpha_{n+1})} = g_\alpha \text{ for all } (\alpha, \alpha_{n+1}) \in \mathbb{N}^{n+1},\ |(\alpha, \alpha_{n+1})| = m.$$

On the other hand, given $\mathcal{D} \subseteq \mathbb{R}^n$, consider its homogenization $\breve{\mathcal{D}} \subseteq \mathbb{R}^{n+1}$, defined as

$$\breve{\mathcal{D}} := \left\{ \lambda \begin{bmatrix} x \\ 1 \end{bmatrix} : x \in \mathcal{D},\ \lambda \ge 0 \right\}.$$

Notice that $g \in \mathcal{P}_{n,m}(\mathcal{D}) \Leftrightarrow \breve{g} \in P_{n+1,m}(\breve{\mathcal{D}})$ for any given $g \in \mathcal{H}_{n,m}$. Hence it follows that $\mathcal{P}_{n,m}(\mathcal{D})$ is isomorphic to $P_{n+1,m}(\breve{\mathcal{D}})$.

An advantage of working with cones of forms is the algebraic characterization of the interior of $P_{n,m}(D)$ stated in Observation 1. The analog statement in general fails for cones of polynomials due to the possibility of "zeros at infinity" (see [26]); for example, the polynomial $g(x_1, x_2) = x_1^2 x_2^2 + (x_1 - x_2)^2$ is positive for all nonzero x but $g \notin \mathrm{int}(\mathcal{P}_{2,4}(\mathbb{R}^2))$ because $g(x_1, x_2) - \epsilon \notin \mathcal{P}_{2,4}(\mathbb{R}^2)$ for all $\epsilon > 0$.

3 Positive semidefinite forms in $\mathbb{R}^n$

In this section we concentrate on the cone $P_{n,2m}(\mathbb{R}^n)$, which we shall abbreviate as $P_{n,2m}$. The inclusion $\Sigma_{n,2m} \subseteq P_{n,2m}$ gives an inner approximation of the cone $P_{n,2m}$ and henceforth a sufficient condition for positive semidefiniteness: a form is psd if it is a sos.

Using $\Sigma_{n,2m}$ as a starting point, we next construct a sequence of inner approximations for $P_{n,2m}$. The construction is based on Observations 2 and 3 below, and a key representation theorem for positive definite forms due to Reznick [27].

3.1 Sums of squares and positive semidefinite matrices

Notice that for $\phi \in H_{n,m}$ we have

$$\phi(x)^2 = \langle \phi, \sigma^m(x) \rangle^2 = \sigma^m(x)^{\mathrm{T}}(\phi\phi^{\mathrm{T}})\sigma^m(x).$$

This yields the following observation.

Observation 2 *Let $\theta \in H_{n,2m}$. Then $\theta \in \Sigma_{n,2m}$ if and only if there exists $\Phi \in \mathcal{S}^{n_m}$, $\Phi \succeq 0$ such that $\theta(x) = \sigma^m(x)^{\mathrm{T}}\Phi\sigma^m(x)$.*

Notice that the identity $\theta(x) = \sigma^m(x)^{\mathrm{T}}\Phi\,\sigma^m(x)$ corresponds to a linear system of equations in Φ and θ.

The embedding operation described next allows us to characterize the set $P_{n,2m} \subseteq H_{n,2m}$ in terms of subsets of $H_{n,2(m+r)}, r = 0, 1, \ldots$.

Observation 3 *Let $\theta \in H_{n,2m}$. Then $\theta \in P_{n,2m} \Leftrightarrow \sum_{j=1}^n x_j^2\,\theta(x) \in P_{n,2m+2}$, and $\theta \in \Sigma_{n,2m} \Rightarrow \sum_{j=1}^n x_j^2\,\theta(x) \in \Sigma_{n,2m+2}$. In particular, if $\sum_{j=1}^n x_j^2\,\theta(x) \in \Sigma_{n,2m+2}$ then $\theta(x) \in P_{n,2m}$.*

3.2 Inner approximations for $P_{n,2m}$

The sequence of cones $K^r_{n,2m}(\mathrm{I\!R}^n)$, $r = 0, 1, \ldots$ defined below gives a sequence of inner approximations for $P_{n,2m}$.

For $r = 0, 1, \ldots$ define

$$\begin{aligned} K^r_{n,2m}(\mathrm{I\!R}^n) \;&:=\; \left\{\theta \in H_{n,2m} : \;\; \left(\textstyle\sum_{j=1}^n x_j^2\right)^r \theta(x) \in \Sigma_{n,2(m+r)}\right\} \\ &=\; \Big\{\theta \in H_{n,2m} : \;\; \exists\, \Phi \in \mathcal{S}^{n_{m+r}},\ \Phi \succeq 0 \text{ s.t} \\ &\qquad\qquad \left(\textstyle\sum_{j=1}^n x_j^2\right)^r \theta(x) = \sigma^{m+r}(x)^{\mathrm{T}}\Phi\sigma^{m+r}(x)\Big\}. \end{aligned}$$

The last identity holds by Observation 2 and automatically gives an LMI description of $K^r_{n,2m}(\mathrm{I\!R}^n)$:

$$K^r_{n,2m}(\mathrm{I\!R}^n) = \{\theta \in H_{n,2m} : \exists\, \Phi \in \mathcal{S}^{n_{m+r}},\ \Phi \succeq 0 \quad \text{s.t } \ \mathcal{L}\theta = \mathcal{T}\Phi\}, \tag{2}$$

where $\mathcal{L} : H_{n,2m} \to H_{n,2(m+r)}$ and $\mathcal{T} : \mathcal{S}^{n_{m+r}} \to H_{n,2(m+r)}$ are the linear maps defined by $(\mathcal{L}\theta)(x) := \left(\sum_{j=1}^n x_j^2\right)^r \theta(x)$ and $(\mathcal{T}\Phi)(x) := \sigma^{m+r}(x)^{\mathrm{T}}\Phi\sigma^{m+r}(x)$.

Proposition 1 $K^r_{n,2m}(\mathrm{I\!R}^n) \uparrow P_{n,2m}$.

Proof. This readily follows from Observations 1, 3 and [27, Thm. 3.12]. □

Remark 1 In general the inclusion $\bigcup_{r=0}^{\infty} K_{n,2m}^r(\mathbb{R}^n) \subseteq P_{n,2m}$ in Proposition 1 is strict. For example, it is known (see, e.g., [27]) that the form

$$\theta(x_1, x_2, x_3, x_4) = x_1^2(x_2^2x_3^4 + x_3^2x_4^4 + x_4^2x_2^4 - 3x_2^2x_3^2x_4^2) + x_4^8$$

satisfies $\theta \in P_{4,8}$ but for all r, $(\sum_j x_j^2)^r\theta(x) \notin \Sigma_{4,8+2r}$, i.e., $\theta \notin \bigcup_{r=0}^{\infty} K_{4,8}^r(\mathbb{R}^4)$.

As shown in Example 1 below, Proposition 1 yields an approximation scheme for unconstrained polynomial optimization. This approximation scheme is different from the one proposed by Lasserre in [13]. In particular, there is no need to assume an *a priori* bound on the size of the minimizer of the polynomial.

Example 1 (Unconstrained polynomial optimization) Let $g(x)$ be a given $(2m)$-degree polynomial in n variables (non-necessarily homogeneous) and consider the problem of finding

$$g^* \ := \ \min\{g(x) : x \in \mathbb{R}^n\}. \tag{3}$$

Notice that this problem is equivalent to

$$\begin{array}{ll} \max & l \\ \text{s.t.} & g(x) \geq l \ \ \forall x \in \mathbb{R}^n. \end{array} \tag{4}$$

Observe that the constraint in (4) is equivalent to $g(x)-l \in \mathcal{P}_{n,2m}(\mathbb{R}^n)$. Without loss of generality assume the constant term of $g(x)$ is zero, i.e., $g_{\vec{0}} = 0$. Thus, by homogenizing, we can recast (4) as

$$\begin{array}{ll} \max & -\theta_{(\vec{0},2m)} \\ \text{s.t.} & \theta_{(\alpha,\alpha_{n+1})} = g_\alpha, \ \forall\, |(\alpha,\alpha_{n+1})| = 2m,\ \alpha \neq \vec{0} \\ & \theta \in P_{n+1,2m}(\mathbb{R}^n \times \mathbb{R}_+) = P_{n+1,2m}(\mathbb{R}^{n+1}). \end{array}$$

For each nonnegative integer r, it is natural to consider the "obvious" approximation

$$\begin{array}{ll} \max & -\theta_{(\vec{0},2m)} \\ \text{s.t.} & \theta_{(\alpha,\alpha_{n+1})} = g_\alpha \ \forall\, |(\alpha,\alpha_{n+1})| = 2m,\ \alpha \neq \vec{0} \\ & \theta \in K_{n+1,2m}^r(\mathbb{R}^{n+1}). \end{array}$$

However, if $g(x) - g^*$ fails to be in the interior of $\mathcal{P}_{n,2m}(\mathbb{R}^n)$, then Theorem 3 cannot be applied. To fix this minor problem, let $h(x) := x_1^{2m} + x_2^{2m} + \cdots + x_n^{2m}$ and consider

$$\begin{array}{lll} g^r \ := & \max & -\theta_{(\vec{0},2m)} \\ & \text{s.t.} & \theta_{(\alpha,\alpha_{n+1})} = g_\alpha + \frac{1}{r+1}h_\alpha, \ \forall\, |(\alpha,\alpha_{n+1})| = 2m,\ \alpha \neq \vec{0} \\ & & \theta \in K_{n+1,2m}^r(\mathbb{R}^{n+1}). \end{array}$$

By construction, this is a semidefinite program. Also by Proposition 1 and Theorem 3, $g^r \uparrow g^*$.

4 Copositive forms

In this section we concentrate on the cone $P_{n,m}(\mathbb{R}^n_+)$, which we call the *cone of m-degree copositive forms* in n variables.

We describe two families of inner approximations of $P_{n,m}(\mathbb{R}^n_+)$. The first one is analogous to that of Section 3. The second one is a sequence of polyhedral cones.

4.1 Inner approximations for $P_{n,m}(\mathbb{R}^n_+)$

Let $\mathscr{S} : H_{n,m} \mapsto H_{n,2m}$, be the map $\theta(x) \mapsto \theta(x^2) := \theta(x_1^2, \ldots, x_n^2)$. In other words,

$$[\mathscr{S}\theta]_\alpha = \begin{cases} \theta_{\frac{1}{2}\alpha} & \text{if every } \alpha_i \text{ is even,} \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Since $\theta(x) \geq 0$ for all $x \in \mathbb{R}^n_+$ if and only if $\theta(x^2) \geq 0$ for all $x \in \mathbb{R}^n$, it follows that

$$\theta \in P_{n,m}(\mathbb{R}^n_+) \Leftrightarrow \mathscr{S}\theta \in P_{n,2m}. \tag{6}$$

Inspired by the connection in (6) we define

$$K^r_{n,m}(\mathbb{R}^n_+) := \{\theta \in H_{n,m} : \mathscr{S}\theta \in K^r_{n,2m}(\mathbb{R}^n)\}.$$

From Proposition 1 and (6) it follows that the sequence of cones $K^r_{n,m}(\mathbb{R}^n_+)$ is a sequence of inner approximations of $P_{n,m}(\mathbb{R}^n_+)$.

Proposition 2 $K^r_{n,m}(\mathbb{R}^n_+) \uparrow P_{n,m}(\mathbb{R}^n_+)$.

The particular case $m = 2$ in Proposition 2 yields the hierarchy of sufficient conditions for copositivity of matrices proposed by Parrilo [19]; namely,

> a symmetric matrix $A \in \mathcal{S}^n$ is copositive (i.e., $a(x) := x^{\mathrm{T}} Ax \in P_{n,2}(\mathbb{R}^n_+)$) if the following r-criterion holds
>
> $$\left(\sum_{j=1}^n x_j^2\right)^r (x^2)^{\mathrm{T}} A\, x^2 \in \Sigma_{n,2+2r}. \tag{7}$$

Corollary 1 *The set of matrices whose corresponding quadratic forms are in $K^r_{n,2}(\mathbb{R}^n_+)$ is the same as the set of matrices that satisfies Parrilo's r criterion* (7)*.*

The LMI description (2) for $K^r_{n,2m}(\mathbb{R}^n)$ yields an LMI description for $K^r_{n,m}(\mathbb{R}^n_+)$. However, a more concise description can be obtained via the cone of *elementary copositive forms* $E_{n,m}(\mathbb{R}^n_+) \subseteq P_{n,m}(\mathbb{R}^n_+)$, defined as follows

$$E_{n,m}(\mathbb{R}^n_+) := \{\theta \in H_{n,m} : \mathscr{S}\theta \in \Sigma_{n,2m}\}. \tag{8}$$

This gives an alternative definition of $K^r_{n,m}(\mathbb{R}^n_+)$; namely

$$K^r_{n,m}(\mathbb{R}^n_+) = \left\{ \theta \in H_{n,m} : \left(\sum_{j=1}^n x_j \right)^r \theta(x) \in E_{n,m+r}(\mathbb{R}^n_+) \right\}.$$

There is a natural modification of the construction of $K^r_{n,m}(\mathbb{R}^n_+)$ that yields a *polyhedral* sequence of approximations of $P_{n,m}(\mathbb{R}^n_+)$. Let

$$C^r_{n,m}(\mathbb{R}^n_+) := \left\{ \theta \in H_{n,m} : \left(\sum_{j=1}^n x_j \right)^r \theta(x) \text{ has non-negative coefficients} \right\}.$$

Clearly $C^r_{n,m}(\mathbb{R}^n_+) \subseteq K^r_{n,m}(\mathbb{R}^n_+)$ as any form with non-negative coefficients is in $E_{n,m}(\mathbb{R}^n)$. By construction, $C^r_{n,m}(\mathbb{R}^n_+)$ has an LMI description. Indeed, $C^r_{n,m}(\mathbb{R}^n_+)$ is a polyhedral cone.

The increasing family of cones $C^r_{n,m}(\mathbb{R}^n_+)$ also yields a sequence of inner approximations for $P_{n,m}(\mathbb{R}^n_+)$.

Proposition 3 $C^r_{n,m}(\mathbb{R}^n_+) \uparrow P_{n,m}(\mathbb{R}^n_+)$.

Proof. This follows from Proposition 2 and Pólya's Theorem [7, Thm. 56]. $\square$

Again, as in Proposition 1, the inclusion $\bigcup_{r=0}^\infty C^r_{n,m}(\mathbb{R}^n_+) \subseteq P_{n,m}(\mathbb{R}^n_+)$ is strict in general. For example, the form

$$\theta(x_1, x_2, x_3, x_4) = x_1(x_2x_3^2 + x_3x_4^2 + x_4x_2^2 - 3x_2x_3x_4) + x_4^4$$

satisfies $\theta \in P_{4,4}(\mathbb{R}^4_+)$ but $\theta \notin \bigcup_{r=0}^\infty C^r_{4,4}(\mathbb{R}^4_+)$.

4.2 Alternative characterization of $E_{n,m}(\mathbb{R}^n_+)$

The next proposition yields a characterization of the elementary copositive forms $E_{n,m}(\mathbb{R}^n_+)$ without relying on the operator $\mathscr{S}$. Aside from being more concise, the alternative description for $E_{n,m}(\mathbb{R}^n_+)$ has natural extensions to pointed polyhedral and semialgebraic cones (cf. Sections 5 and 6). This key property will allow us to extend the ideas presented above to more general cones of psd forms.

Proposition 4 *The sets $E_{n,m}(\mathbb{R}^n_+)$ defined above satisfy the following identity*

$$E_{n,m}(\mathbb{R}^n_+) = \operatorname{coco}\{x_{i_1}x_{i_2}\cdots x_{i_k}\psi(x)^2 : \quad m-k \text{ is even, } \psi \in H_{n,(m-k)/2}, \text{ and } i_1, \ldots, i_k \in \{1, \ldots, n\}\}.$$

Proof. The "$\supseteq$" inclusion is obvious. For the reverse inclusion, assume $\theta \in E_{n,m}(\mathbb{R}^n_+)$. Hence $\theta(x^2) = \sum_{i=1}^k \phi_i(x)^2$, for some $\phi_i \in H_{n,m}$, $i = 1, \ldots, k$. Writing each ϕ_i in terms of its monomial expansion we get

$$\phi_i(x)^2 = \sum_{|\alpha|=m} \sum_{|\alpha'|=m} \phi_{i,\alpha}\phi_{i,\alpha'} x^\alpha x^{\alpha'}.$$

Now let $\texttt{par}(\alpha) \in \{0,1\}^n$ be the *vector of parities* of α, defined by: $\texttt{par}(\alpha)_i = 0$ if α_i is even, and $\texttt{par}(\alpha)_i = 1$ otherwise. Since all monomials in $\theta(x^2) = \sum_{i=1}^k \phi_i(x)^2$ only contain variables with even powers, it follows that

$$\begin{aligned}\theta(x^2) &= \sum_{i=1}^k \phi_i(x)^2 \\ &= \sum_{i=1}^k \sum_{\mathbb{P}\in\{0,1\}^n} \sum \{\phi_{i,\alpha}\phi_{i,\alpha'} x^\alpha x^{\alpha'} : \texttt{par}(\alpha) = \texttt{par}(\alpha') = \mathbb{P}, |\alpha| = |\alpha'| = m\}.\end{aligned}$$

(All other terms cancel out.)

Thus,

$$\theta(x^2) = \sum_{i=1}^k \phi_i(x)^2 = \sum_{i=1}^k \sum_{\mathbb{P}\in\{0,1\}^n} \left(\sum\{\phi_{i,\alpha} x^\alpha : \ \texttt{par}(\alpha) = \mathbb{P},\ |\alpha| = m \ \}\right)^2.$$

Since $\alpha \geq \texttt{par}(\alpha)$, we can rewrite the previous expression as

$$\theta(x^2) = \sum_{i=1}^k \phi_i(x)^2 = \sum_{i=1}^k \sum_{\mathbb{P}\in\{0,1\}^n} x^{2\mathbb{P}} \left(\sum\{\phi_{i,\alpha} x^{\alpha-\mathbb{P}} : \texttt{par}(\alpha) = \mathbb{P}, |\alpha| = m\}\right)^2.$$

Finally since all entries in $\alpha - \texttt{par}(\alpha)$ are even, we get

$$\theta(x) = \sum_{i=1}^k \sum_{\mathbb{P}\in\{0,1\}^n} x^{\mathbb{P}} \left(\sum\{\phi_{i,\alpha} x^{(\alpha-\mathbb{P})/2} : \texttt{par}(\alpha) = \mathbb{P}, |\alpha| = m\}\right)^2.$$

So we are done as each term $x^{\mathbb{P}}(\sum\{\phi_{i,\alpha} x^{(\alpha-\mathbb{P})/2} : \texttt{par}(\alpha) = \mathbb{P}, |\alpha| = m\})^2$ is of the form $x_{i_1}x_{i_2}\cdots x_{i_k}\psi(x)^2$ with $\psi \in H_{n,\frac{m-k}{2}}$. $\square$

The following inductive and LMI descriptions of the sets $E_{n,m}(\mathbb{R}^n_+)$ readily follow from Proposition 4 and Observation 2.

Corollary 2

(i) *The sets $E_{n,m}(\mathbb{R}^n_+)$ satisfy the following recursive relationships*

$$\begin{aligned}
&E_{n,1}(\mathbb{R}^n_+) = \mathrm{conv}\{x_j : j = 1,\ldots,n\},\\
&E_{n,2}(\mathbb{R}^n_+) = \mathrm{conv}(\{(a^{\mathrm{T}}x)^2 : a \in \mathbb{R}^n\} \cup \{x_i x_j : 1 \leq i < j \leq n\}),\\
&E_{n,2k+1}(\mathbb{R}^n_+) = \mathrm{conv}\{x_j\theta(x) : \theta \in E_{n,2k},\ j = 1,\ldots,n\},\\
&E_{n,2k+2}(\mathbb{R}^n_+) = \mathrm{conv}(\Sigma_{n,2(k+1)} \cup \{x_j\theta(x) : \theta \in E_{n,2k+1},\ j = 1,\ldots,n\}).
\end{aligned}$$

(ii) *The sets $E_{n,m}(\mathbb{R}^n)$ can be defined via the following LMI identities*

$$\begin{aligned}
E_{n,1}(\mathbb{R}^n_+) &:= \{\theta \in H_{n,1} : \ \exists a \in \mathbb{R}^n_+ \text{ s.t. } \theta(x) = a^{\mathrm{T}}x\},\\
E_{n,2}(\mathbb{R}^n_+) &:= \{\theta \in H_{n,2} : \ \exists M, N \in \mathcal{S}^n,\ M \succeq 0,\ N \geq 0 \text{ s.t. } \\
&\qquad \theta(x) = x^{\mathrm{T}}(M+N)x\},\\
E_{n,3}(\mathbb{R}^n_+) &:= \{\theta \in H_{n,3} : \ \exists M^i, N^i \in \mathcal{S}^n,\ M^i \succeq 0,\ N^i \geq 0,\ i = 1,\ldots,n \text{ s.t. }\\
&\qquad \theta(x) = \textstyle\sum_i x_i(x^{\mathrm{T}}(M^i + N^i)x)\},
\end{aligned}$$

$$\vdots$$

The map $\mathscr{S}$ defined in (5) establishes a nice parallel between the pairs $(\Sigma_{n,2m}, P_{n,2m})$ and $(E_{n,m}(\mathbb{R}^n_+), P_{n,m}(\mathbb{R}^n_+))$. Extending this parallel, we note that the inclusion $E_{n,m}(\mathbb{R}^n) \subseteq P_{n,m}(\mathbb{R}^n_+)$ is proper except for the special cases described in Proposition 5. This result follows from Theorem 1 (Hilbert's Theorem) and a classical result on copositive forms due to Diananda [4, Thm. 2]. For details see [34].

Proposition 5 *$E_{n,m}(\mathbb{R}^n) = P_{n,m}(\mathbb{R}^n_+)$ if and only $n \leq 2$, $m = 1$, $(n,m) = (3,2)$, or $(n,m) = (4,2)$.*

The LMI description of the sets $E_{n,m}(\mathbb{R}^n_+)$ in Corollary 2 yields an LMI formulation of Parrilo's r-criterion for $r = 0, 1, 2, 3, \ldots$ In particular, we obtain a new succinct derivation of the criterion (9) below for copositivity, as proposed by Parrilo in [19]. Incidentally, Parrilo [19] only showed that (9) implies (7) for $r = 1$. The converse is also true and a proof was given by Bomze and de Klerk in [2].

Proposition 6 *Let $A \in \mathcal{S}^n$ be given. Then*

(i) *A satisfies Parrilo's 0-criterion if and only if there exist $M, N \in \mathcal{S}^n$ with $M \succeq 0$, $N \geq 0$ such that $A = M + N$.*

(ii) *A satisfies Parrilo's 1-criterion if and only if there exist $M^i \succeq 0$, $N^i \geq 0$, $i = 1, \ldots, n$ such that*

$$\sum_{j=1}^{n} x_j (x^{\mathrm{T}} A x) = \sum_{i=1}^{n} x_i (x^{\mathrm{T}} (M^i + N^i) x).$$

Equivalently, if and only if there exist $\Lambda^i \in \mathcal{S}^n$, $i = 1, \ldots, n$ such that

$$\begin{aligned} A - \Lambda^i &\succeq 0, \quad i = 1, \ldots, n \\ \Lambda^i_{ii} &= 0, \quad i = 1, \ldots, n \\ \Lambda^i_{jj} + \Lambda^j_{ji} + \Lambda^j_{ij} &= 0, \quad i \neq j \\ \Lambda^i_{jk} + \Lambda^j_{ik} + \Lambda^k_{ij} &\geq 0, \quad i, j, k \ \text{ all different.} \end{aligned} \tag{9}$$

(iii) *A satisfies Parrilo's 2-criterion if and only if there exist $M \in \mathcal{S}^{n_2}$, $M^{ij}, N^{ij} \in \mathcal{S}^n$, $M \succeq 0$, $M^{ij} \succeq 0$, $N^{ij} \geq 0$, $1 \leq i, j \leq n$ such that*

$$\sum_{k,l} x_k x_l (x^{\mathrm{T}} A x) = \sigma^2(x)^{\mathrm{T}} M \sigma^2(x) + \sum_{i,j} x_i x_j (x^{\mathrm{T}} (M^{ij} + N^{ij}) x).$$

$$\vdots$$

Proof.

(i) By Corollary 1, A satisfies Parrilo's 0-criterion if and only if $A \in K^0_{n,2}(\mathbb{R}^n_+) = E_{n,2}(\mathbb{R}^n_+)$. By Corollary 2, $A \in E_{n,2}(\mathbb{R}^n_+)$ if and only if there exist $M \succeq 0$, $N \geq 0$ such that $A = M + N$.

(ii) Again by Corollary 1, A satisfies Parrilo's 1-criterion if and only if $A \in K^1_{n,2}(\mathbb{R}^n_+)$, that is, if and only if the form $(\sum_j x_j)x^{\mathrm{T}}Ax = \sum_j x_j(x^{\mathrm{T}}Ax)$ belongs to $E_{n,3}(\mathbb{R}^n_+)$. By Corollary 2, $\sum_j x_j(x^{\mathrm{T}}Ax) \in E_{n,3}(\mathbb{R}^n_+)$ if and only if there exist $M^i \succeq 0$, $N^i \geq 0$, $i = 1, \ldots, n$ such that

$$\sum_{j=1}^{n} x_j(x^{\mathrm{T}}Ax) = \sum_{i=1}^{n} x_i(x^{\mathrm{T}}(M^i + N^i)x). \tag{10}$$

To finish we just need to show that (10) is equivalent to (9).

Assume (9) holds, then (10) holds with $M^i = A - \Lambda^i$ and N^i defined componentwise as $N^i_{jk} := \frac{1}{3}(\Lambda^i_{jk} + \Lambda^j_{ki} + \Lambda^k_{ij})$.

Conversely, assume (10) holds. Without loss of generality assume $N^i_{jk} = 0$ unless i, j, k are all different (the M's and N's can always be adjusted to make this hold by increasing some diagonal components on some of the M's). It follows that (9) holds with $\Lambda^i := A - M^i$, $i = 1, \ldots, n$.

(iii) This follows from Corollaries 1 and 2 exactly as in (i) and (ii) above. □

5 Psd forms over pointed polyhedral cones

We next construct approximation schemes for the cone $P_{n,m}(D)$ in the case when D is a pointed polyhedral cone. Throughout this section we shall assume that the domain D is the polyhedral cone

$$D = \{x : a_i^{\mathrm{T}}x \geq 0,\ i = 1, \ldots, q\},$$

for some matrix $\begin{bmatrix} a_1 & \ldots & a_q \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^{q \times n}$. We shall also assume that D is *pointed*, i.e., it contains no lines. It is easy to see that the latter condition is equivalent to $\operatorname{rank}\left(\begin{bmatrix} a_1 & \ldots & a_q \end{bmatrix}^{\mathrm{T}}\right) = n$.

We can extend the ideas of Section 4 in a natural fashion. Let

$$\begin{aligned} E_{n,m}(D) := \operatorname{conv}\{(a_{i_1}^{\mathrm{T}}x) \cdots (a_{i_k}^{\mathrm{T}}x)\,\psi(x)^2 :\quad & m-k \text{ is even, } \psi \in H_{n,(m-k)/2}, \\ & \text{and } i_1, \ldots, i_k \in \{1, \ldots, q\}\}, \end{aligned}$$

$$K^r_{n,m}(D) := \left\{\theta \in H_{n,m} : \left(\sum_{i=1}^{q} a_i^{\mathrm{T}}x\right)^r \theta(x) \in E_{n,m+r}(D)\right\},$$

and

$$\begin{aligned} C^r_{n,m}(D) := \Big\{\theta \in H_{n,m} :\quad & \exists \phi \in H_{n,m+r},\ \phi \geq 0 \text{ s.t.} \\ & \left(\textstyle\sum_{i=1}^{q} a_i^{\mathrm{T}}x\right)^r \theta(x) = \phi(a_1^{\mathrm{T}}x, \ldots, a_q^{\mathrm{T}}x)\Big\}. \end{aligned}$$

Here $\phi \geq 0$ means that the form ϕ has non-negative coefficients.

By construction, both $K_{n,m}^r(D)$ and $C_{n,m}^r(D)$ have LMI descriptions. Indeed, $C_{n,m}^r(D)$ is a polyhedral cone.

The natural extensions of Propositions 2 and 3 hold.

Proposition 7 $C_{n,m}^r(D) \uparrow P_{n,m}(D)$ and $K_{n,m}^r(D) \uparrow P_{n,m}(D)$.

Proof. The first claim follows from Proposition 8 below. The second claim follows from the first one and the inclusions $C_{n,m}^r(D) \subseteq K_{n,m}^r(D) \subseteq P_{n,m}(D)$. $\square$

The following proposition is a generalization of Pólya's Theorem, and can be obtained as a consequence of a representation theorem for polynomials positive on compact polyhedra due to Handelman [5] (a constructive proof of such theorem is presented in [27, Thm. 2]). Here, we provide a proof of this result that relies exclusively on elementary tools and Pólya's Theorem.

Proposition 8 *Assume $D = \{x : a_i^{\mathrm{T}} x \geq 0,\ i = 1, \ldots, q\}$ is pointed and $\theta \in \mathrm{int}(P_{n,m}(D))$. Then for N sufficiently large*

$$(a_1^{\mathrm{T}} x + \ldots + a_q^{\mathrm{T}} x)^N \theta(x) = \phi(a_1^{\mathrm{T}} x, \ldots, a_q^{\mathrm{T}} x)$$

for some $\phi \in H_{q,m+N}$ with $\phi \geq 0$.

Proof. Let $A = QR$ be the full QR-factorization of A (see, e.g., [6, 32]), i.e., $Q \in \mathbb{R}^{q\times q}$ is orthogonal and $R \in \mathbb{R}^{q\times n}$ is upper triangular. Since $\mathrm{rank}(A) = n$ (as D is pointed), the matrix R is of the form $\begin{bmatrix} U \\ 0 \end{bmatrix}$ where $U \in \mathbb{R}^{n\times n}$ is upper triangular and non-singular. Put $Q = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix}$ where Q_1 is the block of the first n columns of Q. Now let $\gamma \in H_{q,m}$ be defined as

$$\gamma(y) := \theta(U^{-1} Q_1^{\mathrm{T}} y).$$

Notice that $U^{-1} Q_1^{\mathrm{T}} A = I$, so in particular $\gamma(Ax) = \theta(x)$.

Claim: For $c \geq 0$ sufficiently large $\gamma(y) + c(y^{\mathrm{T}} Q_2 Q_2^{\mathrm{T}} y) > 0$ for all $y \in \Delta_q := \{y \in \mathbb{R}_+^q : \sum y_i = 1\}$. Here is a proof of the claim: let $F := \{y \in \Delta_q : \gamma(y) \leq 0\}$. If $F = \emptyset$ then take $c = 0$. Otherwise, observe that any given $y \in F$ cannot be of the form Ax, so it is neither of the form $Q_1 z$. Therefore $Q_2^{\mathrm{T}} y \neq 0$ for all $y \in F$. Let $m_1 := \min\{\gamma(y) : y \in F\} \leq 0$, and $m_2 := \min\{(y^{\mathrm{T}} Q_2 Q_2^{\mathrm{T}} y) : y \in F\} > 0$ (these minima are attained because F is compact). The claim then follows by taking $c > -m_1/m_2$.

Let $c > 0$ be sufficiently large so that $\gamma(y) + c(y^{\mathrm{T}} Q_2 Q_2^{\mathrm{T}} y) > 0$ for all $y \in \Delta_q$. Applying Pólya's Theorem to

$$\gamma(y) + c(y_1 + \ldots + y_q)^{m-2}(y^{\mathrm{T}} Q_2 Q_2^{\mathrm{T}} y)$$

we conclude that for N sufficiently large there exists $\phi \in H_{q,m+N}$, $\phi \geq 0$ such that

$$(y_1 + \ldots + y_q)^N(\gamma(y) + c(y_1 + \ldots + y_q)^{m-2}(y^{\mathrm{T}} Q_2 Q_2^{\mathrm{T}} y)) = \phi(y_1, \ldots, y_q).$$

Thus, since $\gamma(Ax) = \theta(x)$ and $Q_2^{\mathrm{T}} Ax = 0$, plugging $y = Ax$ we get

$$(a_1^{\mathrm{T}} x + \ldots + a_q^{\mathrm{T}} x)^N \theta(x) = \phi(a_1^{\mathrm{T}} x, \ldots, a_q^{\mathrm{T}} x).$$

□

6 Psd forms over pointed semialgebraic cones

Consider a domain of the form

$$D = \{x \in \mathbb{R}^n : \phi_i(x) \geq 0,\ i = 1, \ldots, q\},$$

where $\phi_i \in H_{n,m_i}$, $i = 1, \ldots, q$. We shall restrict our attention to domains that are pointed, i.e., we shall assume that 0 cannot be obtained as a sum of nonzero elements of D.

A closed cone D is pointed if and only if $\{0\}$ is an *exposed face* of D (see, e.g., [10, 28]), i.e., if and only if there exists a nonzero vector $a \in \mathbb{R}^n$ such that

$$D \subseteq \{x \in \mathbb{R}^n : a^{\mathrm{T}} x \geq 0\} \tag{11}$$

and

$$D \cap \{x \in \mathbb{R}^n : a^{\mathrm{T}} x = 0\} = \{0\}. \tag{12}$$

Furthermore, if D is a closed cone, then (11) and (12) imply that $D \cap \{x \in \mathbb{R}^n : a^{\mathrm{T}} x = 1\}$ is compact. This holds in particular when D is a semialgebraic cone.

Assumption 1 Assume that a vector $a \in \mathbb{R}^n$ satisfying (11) and (12) is available.

We now extend the constructions in Sections 4 and 5 in a natural fashion. Let

$$E_{n,m}(D) := \mathrm{conv}\{\phi_{i_1}(x) \cdots \phi_{i_k}(x)\, \psi(x)^2 : \begin{array}{l} m - (m_{i_1} + \cdots + m_{i_k}) \text{ is even}, \\ \psi \in H_{n,(m-(m_{i_1}+\cdots+m_{i_k}))/2}, \\ \text{and } i_1, \ldots, i_k \in \{1, \ldots, q\}\}, \end{array} \tag{13}$$

and

$$K_{n,m}^r(D) := \{\theta \in H_{n,m} : (a^{\mathrm{T}} x)^r \theta(x) \in E_{n,m+r}(D)\}. \tag{14}$$

The natural extension of Proposition 2 holds.

Proposition 9 $K_{n,m}^r(D) \uparrow P_{n,m}(D)$.

Proof. First assume $a = e_n := \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^n$. Then the statement above readily follows from a Corollary of Schmüdgen's Theorem [29, Cor. 3]. The general case can be reduced to this special case via a "change of coordinates" as follows. Let $B \in \mathbb{R}^{n\times n}$ be a non-singular matrix whose last column is a. Putting $\tilde{D} := \{y : By \in D\}$, $\tilde{\phi}_i(y) := \phi_i(By)$, $\tilde{a} := e_n$ we are in the previous case and hence the statement above holds for $\tilde{D}$. The general result then follows for D, ϕ_i, a, after changing back to the original coordinates by putting $D = \{By : y \in \tilde{D}\}$, $\phi_i(x) := \tilde{\phi}_i(B^{-1}x)$, $a = B\tilde{a}$. □

Remark 2 Notice that the definitions of $E_{n,m}(\cdot)$ and $K^r_{n,m}(\cdot)$ in Sections 4, 5, and 6 are consistent, i.e., if we apply (13) and (14) to the special cases $D = \mathbb{R}^n_+ = \{x : x_j \geq 0\}$, $a = e := \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}^{\mathrm{T}}$ and to $D = \{x \in \mathbb{R}^n : a_i^{\mathrm{T}} x \geq 0,\ i = 1, \ldots, q\}$, $a = a_1 + \cdots + a_q$, we recover the sets defined in Sections 4 and 5. Indeed, Proposition 2 and the second part of Proposition 7 are special cases of Proposition 9 above.

Using a stronger representation theorem for positive polynomials due to Jacobi and Prestel (see [23, Thm. 6.3.4]), another sequence of inner approximations can be constructed. Assume $m_1, \ldots, m_q$ have the same parity. Let

$$F_{n,m}(D) := \{\phi_i(x)\psi(x)^2 : i \in \{1, \ldots, q\},\ m - m_i \text{ is even, and } \psi \in H_{n,(m-m_i)/2}\},$$

and

$$\mathcal{E}_{n,m}(D) := \operatorname{conv}(F_{n,m}(D) \cup \Sigma_{n,m}),$$

where by convention $\Sigma_{n,m} = \emptyset$ when m is odd.

Finally, let

$$\mathcal{K}^r_{n,m}(D) := \{\theta \in H_{n,m} : (a^{\mathrm{T}}x)^r \theta(x) \in \mathcal{E}_{n,m+r}(D)\}.$$

Proposition 10 $\mathcal{K}^r_{n,m}(D) \uparrow P_{n,m}(D)$.

Proof. This follows from a stronger representation theorem due to Jacobi and Prestel [23, Thm. 6.3.4(i)], proceeding as in the proof of Proposition 9. □

Remark 3 When the m_i's do not have the same parity, the construction above still works provided we take

$$\mathcal{E}_{n,m}(D) := \operatorname{conv}(F_{n,m}(D) \cup G_{n,m}(D) \cup \Sigma_{n,m}),$$

where

$$G_{n,m}(D) := \{\phi_i(x)\phi_j(x)\psi(x)^2 : \begin{array}{l} i < j \in \{1, \ldots, q\}, \\ m - m_i - m_j \text{ is even,} \\ \text{and } \psi \in H_{n,(m-m_i-m_j)/2}\}. \end{array}$$

In this case [23, Thm. 6.3.4(ii)] applies.

Example 2 (Constrained polynomial optimization) Let $g(x)$ and $g_i(x)$, $i = 1,\dots,q$ be given polynomials in n variables (non-necessarily homogeneous) and consider the problem of finding

$$g^* \;:=\; \min\{g(x) : g_i(x) \geq 0,\, i = 1,\dots,q\}. \tag{15}$$

We shall assume that the following technical condition holds:

$$\text{For all } x \in \mathbb{R}^n \setminus \{0\} \text{ there exists } i \in \{1,\dots,q\} \text{ s.t. } \tilde{g}_i(x) < 0, \tag{16}$$

where $\tilde{g}_i(x)$ is the homogeneous component of $g_i(x)$ of highest total degree.

Without loss of generality assume the constant term of $g(x)$ is zero, i.e., $g_{\vec{0}} = 0$. Thus, by homogenizing, it can be shown that (15) is equivalent to

$$\begin{array}{ll} \max & -\theta_{(\vec{0},2m)} \\ \text{s.t.} & \theta_{(\alpha,\alpha_{n+1})} = g_\alpha,\, \forall\, |(\alpha,\alpha_{n+1})| = 2m,\, \alpha \neq \vec{0} \\ & \theta \in P_{n+1,m}(D). \end{array}$$

where $D = \{(x,x_{n+1}) : x_{n+1} \geq 0 \text{ and } \breve{g}_i(x,x_{n+1}) \geq 0,\, i = 1,\dots,q\}$. It is easy to see that the domain $D \subseteq \mathbb{R}^{n+1}$ satisfies Assumption 1 for $a = e_{n+1}$ if and only if condition (16) holds.

For each nonnegative integer r, consider

$$\begin{array}{lll} g^r \;:=& \max & -\theta_{(\vec{0},2m)} \\ & \text{s.t.} & \theta_{(\alpha,\alpha_{n+1})} = g_\alpha,\, \forall\, |(\alpha,\alpha_{n+1})| = 2m,\, \alpha \neq \vec{0} \\ & & \theta \in K^r_{n+1,m}(D). \end{array}$$

By the construction of $K^r_{n+1,m}(D)$ above, this is a semidefinite program. Furthermore, by Proposition 9 and Theorem 3, $g^r \uparrow g^*$. This also holds if $K^r_{n+1,m}(D)$ is changed to $\mathcal{K}^r_{n+1,m}(D)$.

Remark 4 The sequence of inner approximations $\mathcal{K}^r_{n,m}(D)$ is closely related to Lasserre's construction in [13], which relies of a theorem of Putinar [24, Thm. 1.4]. However, as pointed out in [23, page 159], the proof of Putinar's theorem only works in the case when all m_i's are even and requires the hypothesis (16), which is slightly stronger than the hypothesis made in [24] and in [13].

7 Cones of moments

We next study the dual counterpart of cones of positive semidefinite forms; namely, cones of moments.

Let $D \subseteq \mathbb{R}^n$ be a given Borel measurable set, and $\mathbb{B}(D)$ be the set of Borel measures μ in $\mathbb{R}^n$ such that $\mu(D) < \infty$. Define the *cone of moments of order m supported in D* as

$$M_{n,m}(D) := \left\{ y \in H_{n,m} : \exists \mu \in \mathbb{B}(D) \text{ s.t. } y_\alpha = \int_D x^\alpha d\mu(x),\ |\alpha| = m \right\}.$$

The cone $M_{n,m}(D)$ has been extensively studied (see [11]). It is usually referred to as a *moment space*.

It is easy to see that $M_{n,m}(D) = M_{n,m}(\text{cone}(D))$ for any nonempty Borel measurable set $D \subseteq \mathbb{R}^n$. Hence, in the sequel we shall assume without loss of generality that $D \subseteq \mathbb{R}^n$ is a Borel measurable cone.

There is a natural connection between cones of moments and cones of positive semidefinite forms via conic duality. Before presenting this connection we recall the concepts of dual cone and adjoint operator that are crucial for our subsequent discussion (for a detailed discussion see [10, 28]).

Given a cone K in an Euclidean space $(E, \langle\cdot,\cdot\rangle)$, its *dual* is

$$K^* := \{v \in E : \langle u, v\rangle \geq 0 \ \ \forall u \in K\}.$$

Let $T : E \to Y$ be a linear operator between two Euclidean spaces E, Y. The *adjoint* of T is the unique linear operator $T^* : Y \to E$ such that

$$\langle Tu, v\rangle = \langle u, T^*v\rangle \ \ \forall u \in E, \ v \in Y.$$

A straightforward duality argument shows that $P_{n,m}(D) = M_{n,m}(D)^*$. (Recall that $D \subseteq \mathbb{R}^n$ is assumed to be a Borel measurable cone.) This in turn yields

$$\overline{M_{n,m}(D)} = P_{n,m}(D)^* = \overline{\text{conv}\{\sigma^m(x) : x \in D\}},$$

where $\overline{K}$ is the closure of K.

7.1 Cone of moments supported in $\mathbb{R}^n$

In parallel to $P_{n,2m} = P_{n,2m}(\mathbb{R}^n)$, we next study the cone $M_{n,2m}(\mathbb{R}^n)$, which we shall abbreviate as $M_{n,2m}$.

Conic duality and Proposition 1 readily yield ${K^r_{n,2m}}^* \downarrow \overline{M_{n,2m}}$. In other words,

$${K^r_{n,2m}}^* \supseteq {K^{r+1}_{n,2m}}^* \supseteq M_{n,2m}, \text{ and } \bigcap_{r=1}^{\infty} {K^r_{n,2m}}^* = \overline{M_{n,2m}}.$$

The cone ${K^r_{n,2m}}^*$ can also be described via LMI. First, let us recall the following key notation similar to the one in [12]. Let $p \in H_{n,m}$ be given. For $y \in H_{n,m}$ set

$$p(x)|_y := \langle p, y\rangle.$$

In other words, $p(x)|_y$ is the expression obtained when each monomial x^α in $p(x)$ is replaced by y_α. Extend this notation to arrays of m-degree forms componentwise. For example, if

$$g(x) = \begin{bmatrix} x_2^4 & x_1^2x_2^2 & x_1^3x_2 \end{bmatrix}^{\mathrm{T}},$$

then for $y \in H_{2,4}$ we have

$$g(x)|_y = \begin{bmatrix} y_{04} & y_{22} & y_{31} \end{bmatrix}^{\mathrm{T}}.$$

In particular, notice that for all $y \in H_{n,m}$ we have $\sigma^m(x)\big|_y = y$. Notice also that for any given vector $g(x)$ of m-forms, the map $y \mapsto g(x)\big|_y$ is linear.

Define

$$\pi^{2m}(x) := \sigma^m(x)\sigma^m(x)^{\mathrm{T}}.$$

Notice that the entries of the symmetric matrix $\pi^{2m}(x)$ are the same as the entries of $\sigma^{2m}(x)$ (with some repetitions). Likewise, for $y \in H_{n,2m}$ the entries of $\pi^{2m}(x)\big|_y$ are in the same one-to-one correspondence with the entries of y.

For an illustration, suppose $c \in \mathbb{R}^{2m+1}$ and consider the form $y \in H_{2,2m}$ defined by

$$y(x) := c_0 x_1^{2m} + c_1 x_1^{2m-1} x_2 + \cdots + c_{2m-1} x_1 x_2^{2m-1} + c_{2m} x_2^{2m}.$$

In this case $n = 2$, $\sigma^m(x) = \begin{bmatrix} x_1^m & x_1^{m-1}x_2 & \cdots & x_1 x_2^{m-1} & x_2^m \end{bmatrix}^{\mathrm{T}}$, and

$$\pi^{2m}(x)\big|_y = \begin{bmatrix} y_{2m,0} & y_{2m-1,1} & \cdots & y_{m,m} \\ y_{2m-1,1} & y_{2m-2,2} & \cdots & y_{m-1,m+1} \\ \vdots & \vdots & \ddots & \vdots \\ y_{m,m} & y_{m-1,m+1} & \cdots & y_{0,2m} \end{bmatrix} = \begin{bmatrix} c_0 & c_1 & \cdots & c_m \\ c_1 & c_2 & \cdots & c_{m+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_m & c_{m+1} & \cdots & c_{2m} \end{bmatrix}$$

This is the so-called *Hankel matrix* associated to c (see [11]).

For general $y \in H_{n,2m}$, we can think of $\pi^{2m}(x)\big|_y$ as a "generalized Hankel matrix." For example, when $n = m = 3$ and the monomials are ordered lexicographically, we have

$$\sigma^m(x) = \begin{bmatrix} x_1^3 & x_1^2x_2 & x_1^2x_3 & x_1x_2^2 & x_1x_2x_3 & x_1x_3^2 & x_2^3 & x_2^2x_3 & x_2x_3^2 & x_3^3 \end{bmatrix}^{\mathrm{T}}.$$

Hence for $y \in H_{3,6}$ we obtain the following 10×10 matrix

$$\pi^{2m}(x)\big|_y = \begin{bmatrix} y_{600} & y_{510} & y_{501} & & y_{321} & y_{312} & y_{303} \\ y_{510} & y_{501} & y_{420} & \cdots & y_{312} & y_{303} & y_{213} \\ y_{501} & y_{420} & y_{411} & & y_{303} & y_{213} & y_{214} \\ & \vdots & & \ddots & & \vdots & \\ y_{321} & y_{312} & y_{303} & & y_{105} & y_{033} & y_{024} \\ y_{312} & y_{303} & y_{213} & \cdots & y_{033} & y_{024} & y_{015} \\ y_{303} & y_{213} & y_{214} & & y_{024} & y_{015} & y_{006} \end{bmatrix}$$

The generalized Hankel matrices $\pi^{2m}(x)\big|_y$ are denoted by $M(y)$ in [13, 14, 15, 16] and called *moment matrices*. This construction, however, makes sense for any $y \in H_{n,2m}$ regardless of whether y is a vector of moments.

Notice that if $\theta \in H_{n,2m}$ satisfies $\theta(x) = \sigma^m(x)^{\mathrm{T}}\Phi\,\sigma^m(x)$ for some $\Phi \in \mathcal{S}^{n_m}$, then $\theta(x) = \langle \Phi, \pi^{2m}(x)\rangle$ and consequently

$$\langle \theta, y\rangle = \langle \Phi, \pi^{2m}(x)\big|_y\rangle \ \text{ for all } \ y \in H_{n,2m}.$$

Thus, Observation 2 yields

$$\Sigma_{n,2m}{}^* = \{y \in H_{n,2m} : \pi^{2m}(x)\big|_y \succeq 0\}. \tag{17}$$

This identity and the construction of $K^r_{n,2m}$ in turn yield the following LMI characterization of $K^r_{n,2m}{}^*$.

$$K^r_{n,2m}{}^* = \Big\{ y \in H_{n,2m} : \quad \exists y' \in H_{n,2m+2r} \text{ s.t. } \ \pi^{2m+2r}(x)\big|_{y'} \succeq 0 \text{ and } \\ y = \left(\textstyle\sum_{j=1}^n x_j^2\right)^r \sigma^{2m}(x)\Big|_{y'} \Big\}.$$

7.2 Cone of moments supported in pointed cones

We now turn our attention to the cone of moments $M_{n,m}(D)$ for the conic domains D studied in Sections 4, 5, and 6. First, consider the general case, i.e., assume

$$D = \{x \in \mathrm{I\!R}^n : \phi_i(x) \geq 0, \ i = 1, \dots, q\},$$

is such that Assumption 1 holds.

Again conic duality and Proposition 4 yield $K^r_{n,m}(D)^* \downarrow \overline{M_{n,m}(D)}$.

The construction of $K^r_{n,m}(D)$ yields the following characterization of $K^r_{n,m}(D)^*$.

$$K^r_{n,m}(D)^* = \{y \in H_{n,m} : \exists y' \in E_{n,m+r}(D)^* \text{ s.t. } y = (a^{\mathrm{T}}x)^r \sigma^m(x)\big|_{y'}\}. \tag{18}$$

An LMI-description for $K^r_{n,m}(D)^*$ will follow from the LMI-description of $E^r_{n,m}(D)^*$ that we provide in Section 7.3 below. Before doing so, we briefly discuss an alternative polyhedral approximation of $\overline{M_{n,m}(D)}$ for D polyhedral.

When $D = \{x \in \mathrm{I\!R}^n : a_i^{\mathrm{T}} x \geq 0, \ i = 1, \dots, q\}$ conic duality yields the sequence of polyhedral approximations $C^r_{n,m}(D)^* \downarrow \overline{M_{n,m}(D)}$. Each $C^r_{n,m}(D)^*$ in turn can be characterized as

$$C^r_{n,m}(D)^* = \{y \in H_{n,m} : \exists y' \geq 0 \text{ s.t. } y = (a_1^{\mathrm{T}}x + \cdots + a_q^{\mathrm{T}}x)^r \sigma^m(a_1^{\mathrm{T}}x, \dots, a_q^{\mathrm{T}}x)\big|_{y'}\}.$$

7.3 The dual cones $E_{n,m}(D)^*$ and $\mathcal{E}_{n,m}(D)^*$

We next derive a recursive LMI-description of $E_{n,m}(D)^*$ and $\mathcal{E}_{n,m}(D)^*$. These readily follows from duality, and the construction of $E_{n,m}(D)$ and $\mathcal{E}_{n,m}(D)$ in Section 6.

Proposition 11 *Let $y \in H_{n,m}$ be given.*

(i) *Assume m is odd. Then $y \in E_{n,m}(D)^*$ if and only if for each $i = 1, \dots, q$ with $m_i \leq m$, there exists $y^i \in E_{n,m-m_i}(D)^*$ such that*

$$\phi_i(x)\sigma^{m-m_i}(x)\big|_y = y^i. \tag{19}$$

(ii) *Assume m is even. Then $y \in E_{n,m}(D)^*$ if and only if*

$$\pi^{2m}(x)\big|_y \succeq 0,$$

and for each $i = 1, \ldots, q$ with $m_i \le m$, there exists $y^i \in E_{n,m-m_i}(D)^*$ such that (19) *holds.*

Corollary 3 *The cones $E_{n,m}(\mathbb{R}^n_+)^*$ can be defined via the following LMI identities*

$$\begin{aligned}
E_{n,1}(\mathbb{R}^n_+)^* &= \{y \in H_{n,1} : y \ge 0\},\\
E_{n,2}(\mathbb{R}^n_+)^* &= \{y \in H_{n,2} : y \ge 0,\, \pi^2(x)\big|_y \succeq 0\},\\
E_{n,3}(\mathbb{R}^n_+)^* &= \{y \in H_{n,3} : y \ge 0,\, x_i\pi^2(x)\big|_y \succeq 0, i = 1, \ldots, n\},\\
E_{n,4}(\mathbb{R}^n_+)^* &= \{y \in H_{n,4} : y \ge 0,\, \pi^4(x)\big|_y \succeq 0 \;\; x_ix_j\pi^2(x)\big|_y \succeq 0, 1 \le i < j \le n\},\\
&\;\;\vdots
\end{aligned}$$

Proposition 12 *Assume $m_1, \ldots, m_q$ have the same parity, and $y \in H_{n,m}$ is given. Then $y \in \mathcal{E}_{n,m}(D)^*$ if and only if the following two conditions hold:*

(i) *if m has the same parity as $m_1, \ldots, m_q$ then for each $i = 1, \ldots, q$ with $m_i \le m$,*

$$\phi_i(x)\pi^{m-m_i}(x)\big|_y \succeq 0,$$

(ii) *if m is even then*

$$\pi^m(x)\big|_y \succeq 0.$$

Remark 5 If not all $m_1, \ldots, m_q$ have the same parity then, assuming $\mathcal{E}_{n,m}(D)$ is constructed as in Remark 3, the characterization of $\mathcal{E}_{n,m}(D)^*$ in Proposition 12 still holds provided the following condition is added:
(iii) if $m - m_i - m_j$ with $i < j$ is even then

$$\phi_i(x)\phi_j(x)\pi^{m-m_i-m_j}(x)\big|_y \succeq 0.$$

7.4 Non-recursive description of $E_{n,m}(\mathbb{R}^n_+)^*$

As for $E_{n,m}(\mathbb{R}^n_+)$, it is possible to construct a non-recursive description of $E_{n,m}(\mathbb{R}^n_+)^*$ at the expense of increasing the dimension of the problem.

Recall from Section 4 the map $\mathscr{S} : H_{n,m} \mapsto H_{n,2m}$, defined by $\theta(x) \mapsto \theta(x^2)$. It is easy to see that for $\theta, y \in H_{n,m}$

$$\langle \theta, y\rangle = \langle \mathscr{S}\theta, \mathscr{S}y\rangle. \tag{20}$$

We have the following non-recursive LMI description of $E_{n,m}(\mathbb{R}^n_+)^*$.

Proposition 13

$$\begin{aligned}
E_{n,m}(\mathbb{R}^n_+)^* &= \{y \in H_{n,m} : \mathscr{S}y \in \Sigma^*_{n,2m}\}\\
&= \{y \in H_{n,m} : \pi^{2m}(x)\big|_{\mathscr{S}y} \succeq 0\}.
\end{aligned}$$

The proof of Proposition 13 relies on the following result, which is of independent interest.

Lemma 4 *For all $\theta \in \Sigma_{n,2m}$, we have $\mathscr{S}^*\theta \in E_{n,m}(\mathbb{R}^n_+)$.*

Proof. Let $\theta \in \Sigma_{n,2m}$ be given. Put $\theta_0(x) = \theta(x)$, and

$$\theta_i(x) = \frac{1}{2}(\theta_{i-1}(x_1, \ldots, -x_i, \ldots, x_n) + \theta_{i-1}(x)) \ \text{ for } i = 1, \ldots, n.$$

Then $\theta_i \in \Sigma_{n,2m}$ for $i = 0, \ldots, n$ and $\mathscr{S}(\mathscr{S}^*\theta) = \theta_n \in \Sigma_{n,2m}$. Thus by (8), $\mathscr{S}^*\theta \in E_{n,m}(\mathbb{R}^n_+)$. □

Proof of Proposition 13. The first identity from (8), (20) and Lemma 4. The second identity follows from (17). □

Proposition 13 and (18) yield the following LMI characterization of $K^r_{n,m}(\mathbb{R}^n_+)^*$.

$$K^r_{n,m}(\mathbb{R}^n_+)^* = \Big\{ y \in H_{n,m} : \ \exists y' \in H_{n,m+r} \text{ s.t. } \pi^{2m+2r}(x)\big|_{\mathscr{S}y'} \succeq 0 \\ \text{and } y = \left(\textstyle\sum_{j=1}^n x_j\right)^r \sigma^m(x)\Big|_{y'} \Big\}.$$

The LMI characterizations of $E_{n,m}(\mathbb{R}^n_+)^*$ given by Corollary 3 and Proposition 13 are of course equivalent. In principle the latter has the advantage of being non-recursive but the drawback of going through the higher dimensional space $H_{n,2m}$. This can be illustrated by comparing the two LMI-descriptions in a simple example: For $n = 3$, $m = 2$, $r = 0$ the characterization of Proposition 13 yields (with a suitable, though not at all obvious, ordering of the monomials)

$$y = \begin{bmatrix} y_{200} \\ y_{020} \\ y_{002} \\ y_{110} \\ y_{101} \\ y_{011} \end{bmatrix} \in E_{3,2}(\mathbb{R}^n_+)^* \Leftrightarrow \begin{bmatrix} y_{200} & y_{110} & y_{101} & 0 & 0 & 0 \\ y_{110} & y_{020} & y_{011} & 0 & 0 & 0 \\ y_{101} & y_{011} & y_{002} & 0 & 0 & 0 \\ 0 & 0 & 0 & y_{110} & 0 & 0 \\ 0 & 0 & 0 & 0 & y_{101} & 0 \\ 0 & 0 & 0 & 0 & 0 & y_{011} \end{bmatrix} \succeq 0.$$

The latter is equivalent to

$$\begin{bmatrix} y_{200} & y_{110} & y_{101} \\ y_{110} & y_{020} & y_{011} \\ y_{101} & y_{011} & y_{002} \end{bmatrix} \succeq 0 \ \text{ and } \ \begin{bmatrix} y_{200} & y_{110} & y_{101} \\ y_{110} & y_{020} & y_{011} \\ y_{101} & y_{011} & y_{002} \end{bmatrix} \geq 0,$$

i.e.,

$$\pi^2(x)\Big|_y \succeq 0 \ \text{ and } \ y \geq 0,$$

which is precisely the description of $E_{3,2}(\mathbb{R}^n_+)^*$ given by Corollary 3.

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