

State Space Reduction for Non-stationary Stochastic Shortest Path Problems with Real-Time Traffic Information

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Abstract—Routing vehicles based on real-time traffic conditions has been shown to significantly reduce travel time, and hence cost, in high-volume traffic situations. However, taking real-time traffic data and transforming them into optimal route decisions is a computational challenge. This is in large part due to the amount of data available that could be valuable in the route selection. We model the dynamic route determination problem as a Markov decision process (MDP) and present procedures for identifying traffic data having no decision-making value. Such identification can be used to reduce the state space of the MDP thereby improving its computational tractability. This reduction can be achieved by a two-step process. The first is an a priori reduction that may be performed using a stationary, deterministic network with upper and lower bounds on the cost functions before the trip begins. The second part of the process reduces the state space further on the non-stationary stochastic road network as the trip optimally progresses. We demonstrate the potential computational advantages of the introduced methods based on actual data collected on a road network in southeast Michigan.

Index Terms—Markov Decision process, dynamic programming, non-stationary stochastic shortest path problem, state space reduction, vehicle routing, real-time traffic information.

I. INTRODUCTION

RECENTLY, there has been growing interest in determining the value of real-time traffic information for fleet management. Such information can be used in an *intelligent transportation system* (ITS) to reroute vehicles while in-route to avoid areas of high congestion. In an ideal world, all links in a network would be instrumented with traffic sensors (magnetic loops, cameras, etc.) and optimal routes would be computed (and re-computed) in real-time. As a vehicle approaches an intersection, the driver would be given updated directions based on the time varying and random travel time information of the network. Such an implementation requires that a central dispatcher monitor road conditions for the whole network and maintain communication with a vehicle throughout the trip. As the number of links in a network increases, observation

and route determination becomes computationally challenging (cf. Polychronopoulos and Tsitsiklis [1]). In this paper, we note that in some cases, information about the whole network is not required for optimal route determination. For example, in routing a vehicle from the Washington, D.C., Zoo to the Washington, D.C., National Airport, there may be no need to have information about travel congestion on links outside of the Beltway. Our objective then is (for a fixed origin and destination) to find a systematic method for identifying links in a road network that even when observed, do not stand to aid in optimal route determination. We verify via a numerical study that our methods can stand to alleviate some of the computational challenges of real-time route determination in a time varying, stochastic network. We note that this is different from problems in traffic control (or assignment) that attempt to control traffic flows from a system point of view since we are interested in routing a single vehicle through a potentially congested network.

With this in mind, consider the following decision-making scenario; a formal model description will be given in Section II. A vehicle must travel from a start node to a goal node on a network composed of links having (possibly) non-stationary travel times. A subset of these links (called *observed links*) provide real-time traffic information. Depending on the current congestion status of each observed link, the decision-maker must choose the next intersection to visit. Upon reaching the next intersection (or perhaps slightly before), the status of the network is updated, the decision-maker chooses the next node to visit and the trip continues.

The classic shortest path problem has been studied extensively in the literature. When the travel costs are deterministic, a route can be chosen to minimize the total cost before the trip begins. This continues to hold when the travel times are stationary and stochastic since random travel times can be replaced with their expectations. On the other hand, Hall [2] showed that standard shortest path algorithms (such as Dijkstra's algorithm) do not find the minimum expected cost path on a non-stationary, stochastic network. The best route from any given node to the final destination depends not only on the node, but also on the arrival time to that node. Thus, the optimal route choice is not a simple path but a policy that prescribes what node should be visited once the arrival time to a node is realized. This is consistent with the decision-making scenario described in the previous paragraph. Hall [2] suggested dynamic programming for finding the optimal policy.

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When a road map is characterized by a tightly interconnected network of nodes, the complexity of finding the shortest path in a deterministic network was estimated in Pearl [3]. Using the A^* algorithm, guided by the Euclidean heuristic, it was shown that a fraction of the nodes expanded under breadth-first search would also be expanded by A^* . Adaptations of the A^* algorithm for path determination are found in Bander and White [4] and Chabini [5]. AO^* for a non-stationary stochastic shortest path problem with terminal cost was investigated by Bander and White [6] who showed that AO^* is more computationally efficient than dynamic programming when lower bounds on the value function are available. Similar formulations to our model without real-time traffic congestion information were also presented. A hierarchical routing algorithm that finds a near-optimal route with improved computational performance is in Jagadeesh et al. [7]. Miller-Hooks and Mahmassani [8] compared algorithms for the least possible cost path in the discrete-time non-stationary stochastic case. Miller-Hooks and Mahmassani [9] produced an algorithm for finding the least expected cost path in that case.

Of particular relevance to this paper is the *stochastic shortest path problem with recourse* originally studied by Croucher [10]. In this work, when a driver chooses a link to traverse, there is a fixed probability that it will actually traverse an adjacent link as opposed to the one chosen. One might think of the original link as being one on which an accident or unexpected road block has occurred. Interesting extensions of this work include that of Andreatta and Romeo [11] and Polychronopoulos and Tsitsiklis [1]. In the prior, a path from the source to the destination is chosen *a priori*. There is then a fixed probability upon arriving to one of the nodes in the network that the next link is “inactive” and an alternate route must be chosen. In the latter work, the authors assume that the costs to traverse arcs are random variables and that travelling along the network reduces the possibilities for “states” of the network. The problem we consider has a less general cost structure than Polychronopoulos and Tsitsiklis [1] but allows for both the costs and travel times to be non-stationary in time. Moreover, we provide conditions under which the required state observations are reduced. The recent paper by Waller and Ziliaskopoulos [12] addresses the question with random arc costs but with only local arc and time dependencies. In Chabini [13] a solution is provided for the efficient algorithms to compute all-to-one shortest paths in discrete dynamic networks. In Chabini and Ganugapati [14] design, implementation, and computational testing are reported for parallel algorithms that exploit possibilities offered by low-cost, commonly available, parallel, and distributed computing platforms to solve many-to-many shortest path problems in time-dependent networks. Gao and Chabini [15] discuss several approximations of the time dependent stochastic shortest path problem where information about the network is learned by the driver as the trip progresses. Psaraftis and Tsitsiklis [16] considered a problem similar to the one discussed in this paper in which the travel time distributions on links evolve over time according to a Markov process except that the changes in the status of the link are not observed until the vehicle arrives at the link. The work

of Fu and Rilett [17] and Fu [18] discusses implementations of real-time vehicle routing based on estimation of mean and variance travel times. A comprehensive literature review of shortest path problems can be found in Thomas [19]. Kim et al. [20] extended Psaraftis and Tsitsiklis [16] to examine the case where the congestion status of each link is available to the driver. Using actual traffic data in a congested traffic environment in southeast Michigan, the results have shown an average expected cost savings of approximately 5.91-10.55% and a reduction in the vehicle usage time of approximately 9.82-16.19%. We continue to consider this problem in the present study, but the focus is on making the problem more tractable via our state space reduction techniques.

The rest of the paper is organized as follows. Section II presents a formulation of the problem using a stochastic dynamic program. The optimality equations are discussed in Section III. Section IV presents a procedure to reduce the state space for the non-stationary stochastic shortest path problem with real-time traffic congestion information. The first step of this procedure reduces the state space by eliminating observed links that are redundant throughout the trip; that is, those links that would not be traversed by an optimal route regardless of their traffic congestion information. This step can be quickly performed by considering a deterministic road network using upper and lower bounds on the cost functions along each path. The second step of this procedure further reduces the state space by deleting observed links that could not be deleted when the trip began but can be deleted later in the trip. Section V presents computational results based on actual data collected and a road network in southeast Michigan. The results demonstrate (on average) a significant computational advantage when our state space reduction techniques are employed. In Section VI, conclusions and future research directions are discussed. Notation and definitions are listed in the appendix.

II. PROBLEM STATEMENT

We now formulate the non-stationary stochastic shortest path problem with real time traffic information as a discrete-time, finite horizon Markov decision process (MDP). Consider an underlying network $G \equiv (N, A)$, where the finite set N represents the set of nodes and $A \subseteq N \times N$ is the set of directed links in the network. This network serves as a model of a network of roads (links) and intersections (nodes). By this, we mean that $(n, n') \in A$ if and only if there is a road segment that permits traffic to travel from intersection n to intersection n' . Let $n_0 \in N$ be the start node and the set $\Gamma \subseteq N$ be the goal node set. For each element $n \in N$ define the *successor set* of n (denoted $SCS(n)$) to be the set of nodes that have an incoming link emanating from n . That is, $SCS(n) \equiv \{n' : (n, n') \in A\}$. A path $p = (n_0, \dots, n_K)$ from $n_0 \in N$ is a sequence of nodes such that $n_{k+1} \in SCS(n_k)$ for $k = 0, 1, \dots, K - 1$. We assume the existence of a path from any node in the network to the goal node set.

A link $(n, n') \in A$ is said to be *observed* if real-time traffic is measured and reported on (n, n') . Suppose that there are Q observed links in A . Define the (random) road congestion status vector at time t to be $Z(t) = \{Z^1(t), \dots, Z^Q(t)\}$ so

that the random variable $Z^q(t)$ is

$$Z^q(t) = \begin{cases} 0 & \text{if the } q^{\text{th}} \text{ link is uncongested} \\ 1 & \text{if the } q^{\text{th}} \text{ link is congested} \end{cases}$$

for $q = 1, 2, \dots, Q$. Denote a realization of $Z(t)$ by z . Thus, $z \in H \equiv \{0, 1\}^Q$. We assume that $\{Z^i(t), t = t_0, t_0 + 1, \dots\}$ and $\{Z^j(t), t = t_0, t_0 + 1, \dots\}$ are independent Markov chains for $i \neq j$. For each $q = 1, 2, \dots, Q$, we assume that the dynamics of the corresponding Markov chain are described by the one-step transition matrix

$$R_q^{(t,t+1)} = \begin{bmatrix} \alpha_t^q & 1 - \alpha_t^q \\ 1 - \beta_t^q & \beta_t^q \end{bmatrix}.$$

Let $P(z'|t, z, t')$ be the probability of a transition occurring from z at time t to z' at time t' . By our assumption of independence,

$$P(z'|t, z, t') \equiv P[Z(t') = z' | Z(t) = z] \\ = \prod_{q=1}^Q P[Z^q(t') = (z^q)' | Z^q(t) = z^q]$$

where $(z^q)'$ is the q^{th} element of z' and each term satisfies a simple extension of Kolmogorov equations for the non-stationary case (cf. Theorem 5.4.3 of Ross [21])

$$R_q^{(t,t')} = \begin{bmatrix} \alpha_t^q & 1 - \alpha_t^q \\ 1 - \beta_t^q & \beta_t^q \end{bmatrix} \times \begin{bmatrix} \alpha_{t+1}^q & 1 - \alpha_{t+1}^q \\ 1 - \beta_{t+1}^q & \beta_{t+1}^q \end{bmatrix} \\ \times \dots \times \begin{bmatrix} \alpha_{t'}^q & 1 - \alpha_{t'}^q \\ 1 - \beta_{t'}^q & \beta_{t'}^q \end{bmatrix}.$$

Thus, the congestion status of the network is modeled by a Q -dimensional, non-homogeneous Markov chain.

Let $P(t'|n, t, z, n')$ denote the probability of arriving at node n' at time t' , given that the vehicle travels from node n to n' , departing node n at time t with congestion status vector z . This probability distribution is assumed to be discrete, and the travel time between any two nodes is assumed to be bounded, i.e., $P(t'|n, t, z, n') = 0$ for all $t' \leq t$ and for all $t' > t + \zeta$ for a given, finite value ζ . Notice that we have not precluded n from being a member of its own successor set. When this situation occurs, the driver is allowed to wait at node n , and we assume that $P(t+1|n, t, z, n) = 1$, for all t and z .

Denote the k^{th} node visited by the vehicle as n_k , and let t_k be the time that node n_k is visited. Thus, n_0 is the start node and t_0 the start time. We assume the existence of a fixed (and known) time T after which no other decisions are made. Thus, if n_k is the k^{th} node visited and t_k is the time of the k^{th} decision epoch, upon arrival to the next node, say n_{k+1} at time $t_{k+1} > T$, a terminal cost $\bar{c}(n_{k+1}, t_{k+1})$ is accrued. We assume that $\bar{c}(n, t) = \infty$ for all $t > T$ and $n \notin \Gamma$.

Since the goal of a decision-maker is to find the minimum cost path, T is simply for modelling convenience and can be chosen to be arbitrarily large. We choose T such that if t_0 is less than some $\hat{T} \ll T$ there exists a path from any node to Γ such that the time of arrival to Γ is (guaranteed) to be less than T . This is possible as a consequence of the bounded travel times assumption. For example, suppose ζ bounds the travel times on each link. Recall that we have assumed the

existence of a path from any node to Γ . Since $|N| < \infty$ these paths can be chosen to be finite. Choosing T and \hat{T} such that $T - \hat{T} > |N|\zeta$ guarantees our assertion.

Let $U = \{0, 1, \dots, T\}$ be the set of possible times that decisions are made. Define the state space of our decision problem to be

$$\Omega \equiv \{(n, t, z) : n \in N, t \in U, z \in H\}.$$

A (deterministic, Markov) policy π is a function such that $\pi : \Omega \rightarrow N$ and prescribes which node should be visited next for each node, time, and congestion status; that is, $\pi(n, t, z) \in SCS(n)$. Let z_k be the congestion status at time t_k . We assume for $n_k \in \Gamma$ that $\pi(n_k, t_k, z_k) \in \Gamma$ for all π ; i.e., once the goal node set is reached, it is never left. When the decision is made to travel from n_k to $n_{k+1} = \pi(n_k, t_k, z_k)$, the (finite) random variable T_{k+1} is determined according to the discrete probability distribution $P[T_{k+1} = t_{k+1} | n_k, t_k, z_k, n_{k+1}]$.

Let $\tilde{c}(n, t, z, n', t')$ be the cost accrued by traversing road segment $(n, n') \in A$, given that travel begins at time t and ends at time t' with the congestion status z at time t . We assume the congestion of other links does not affect the cost of traversing, or the time to traverse, the current link. For $n \in \Gamma$ assume that $\tilde{c}(n, t, z, n', t') = \bar{c}(n, t) = 0$. Thus, once the goal node set is reached, no other costs are accrued. Define the expected cost of traveling from n to n' , starting at time t given congestion status z , by

$$c(n, t, z, n') = \sum_{t'} P(t'|n, t, z, n') \tilde{c}(n, t, z, n', t').$$

Note that in many practical applications, there are ‘‘ideal’’ and ‘‘latest acceptable’’ arrival time windows to the goal node set. Thus, for a given link (n, n') such that $n \notin \Gamma$ and $n' \in \Gamma$ there may be two components of the cost function $\tilde{c}(n, t, z, n', t')$, a part corresponding to the cost of traversing the link (n, n') and a part corresponding to arriving at the goal node set at time t' that represents the penalty for late or early arrival. Assume for all $(n, t, z) \in \Omega$ and n' such that $n \notin \Gamma$ and $t' \leq T$, there exists $b, B \in \mathbb{R}^+$ such that $0 < b \leq c(n, t, z, n', t') \leq B < \infty$. We can now define the total expected cost of a policy.

Let the random variable M be the number of decision epochs before time T . The total expected cost accrued under policy π is

$$v^\pi(n_0, t_0, z_0) = \mathbb{E}_{n_0, t_0, z_0}^\pi \left\{ \sum_{k=0}^M c[n_k, t_k, z_k, \pi(n_k, t_k, z_k)] + \bar{c}(n_{M+1}, t_{M+1}) \right\}$$

where $\mathbb{E}_{n_0, t_0, z_0}^\pi$ is the expectation operator, conditioned on beginning at state $(n_0, t_0, z_0) \in \Omega$. We seek an optimal policy π^* such that

$$v^*(n_0, t_0, z_0) \equiv v^{\pi^*}(n_0, t_0, z_0) \leq v^\pi(n_0, t_0, z_0),$$

for all policies π .

III. OPTIMALITY EQUATIONS AND PRELIMINARY RESULTS

For any function $f : \Omega \rightarrow \mathbb{R}^+$, let

$$h[n, t, z, n', f] \equiv c[n, t, z, n'] + \sum_{t'} P[t'|n, t, z, n'] \sum_{z'} P[z'|t, z, t'] f[n', t', z'].$$

Thus, h represents the total expected cost when in state (n, t, z) , n' is chosen as the next node to visit and a terminal reward f is accrued after moving to n' .

For any $(n, t, z) \in \Omega$ we call the following the *optimality equations*

$$v^*(n, t, z) = \min_{n' \in SCS(n)} \{h(n, t, z, n', v^*)\} \quad (1)$$

subject to the boundary conditions $v^*(n, t, z) = \bar{c}(n, t)$ for $t > T$. Furthermore, a policy, π^* , is optimal if and only if, for all n, t, z ,

$$\pi^*(n, t, z) \in \arg \min_{n' \in SCS(n)} \{h(n, t, z, n', v^*)\}. \quad (2)$$

When considered together, the boundedness of the cost function $c(n, t, z, n', t')$ (for $t' \leq T$), the existence of a path from each node to Γ and the fact that $|N| < \infty$ guarantee the existence of an optimal policy with finite expected cost for any initial time such that the goal node is reached before T .

Since the problem posed is a finite state and action space, finite horizon MDP, a simple extension of the results of Chapter 4 of Puterman [22] yields that the value function related to the non-stationary stochastic shortest path problem with real time congestion information satisfies the optimality equation (1), that (2) characterizes an optimal policy, and that we can use (1) to compute v^* via backward induction.

As explained in Kim [23], the size of the state space is $|N| \times T \times 2^Q$, i.e., the number of nodes times the number of time units times the number of states in the Markov chain for the observed links. Since the state space can be quite large for large Q , solving the optimality equations recursively may be impractical. The remainder of the paper is devoted to easing this burden. To this end, we remark that if we divide the vector of observed links, z , into two sub-vectors such that $z = (\tilde{z}, \hat{z})$ and,

$$v^*(n, t, (\tilde{z}, \hat{z})) = v^*(n, t, \tilde{z}),$$

for all z , then for node n and time t we can eliminate \hat{z} from our state space without loss of optimality. Thus, observing the links corresponding to \hat{z} at time t does not improve the optimal total cost. Furthermore, for $t' > t$ we hope that this continues to be true for every node in the optimal path. Hence, the objectives of our state space reduction algorithm are:

- To identify what elements in the z -vector can be eliminated before the start of the trip.
- To find conditions that guarantee the eliminated elements of z remain eliminated throughout the search for the optimal policy without loss of optimality.
- To develop quick and efficient techniques for z -vector reduction.
- To implement these techniques in practice for real-time vehicle routing.

In the next section we begin to address these concerns.

IV. STATE SPACE REDUCTION

A. A priori state space reduction

In this section we describe a procedure for state space reduction before the trip begins. The motivation is that when the driver travels from the origin to the destination, observing some links does not improve the optimal total cost throughout the trip. Indeed, some links would not be traversed even if the usual route taken by the driver is congested. This situation is illustrated in Figure 1. The small marks indicate observed links. Those outside of the dashed oval are not required to determine an optimal route and simply add superfluous information. That is to say, solving the non-stationary, stochastic shortest path problem with the 10 observed links on the left, yields the same optimal expected total cost as that on the right with only 4 observed links. Unfortunately, identifying observed links that are superfluous for a particular trip is not always intuitive in an urban network of roads.

In order to obtain the main results of this section, we bound the costs of traversing observed links. This allows us to bound the effect that each link can have on the optimal cost function. If this effect is zero, we need not consider the congestion status of the link and it can be deleted from the state space. We assume throughout the remainder of this section that the goal node set $\Gamma = \{\gamma\}$ is a singleton. This is done for simplicity only since the results can be extended to the more general case where Γ is a subset of N .

Fix $t_0 < \hat{T}$ and recall that this implies that there exists a path from any node to γ such that the time of arrival to γ is (guaranteed) to be less than T . Thus, for the remainder of the paper, assume that all times are less than T so that the all expected costs are finite. Let $\underline{c}(n, t, n')$ and $\bar{c}(n, t, n')$ be the expected link cost from n to n' when the observed link (n, n') is uncongested and congested, respectively, at time t . Define $\underline{c}(n, n')$ and $\bar{c}(n, n')$ as follows:

$$\underline{c}(n, n') \equiv \min_s \{c(n, s, n')\} \\ \bar{c}(n, n') \equiv \max_s \{\bar{c}(n, s, n')\}$$

for each observed link $(n, n') \in A$.

Suppose a path from node n to γ , denoted $F(n, \gamma)$, is chosen by historical data or by previous experience. It is often the case that a driver has made the trip from n to γ regularly and has a preferred path that can be used in this step. Let $\bar{u}(n, t)$ be the expected total cost from n at time t to γ along the predetermined path when all observed links in the path are deterministically congested. Define similarly $\bar{u}(n)$ to be the total cost from n to γ along the predetermined path when $\bar{c}(n, n')$ is assigned to each observed link (n, n') in the path for all $t \leq T$. Furthermore, let $\underline{v}(n, t)$ be the minimum expected total cost from n at time t to γ when all observed links are deterministically uncongested and $\underline{v}(n)$ be the minimum total cost from n to γ when $\underline{c}(n, n')$ is assigned to each observed link $(n, n') \in A$ for all $t \leq T$. Define $\underline{v}(n, t, (m, m'))$ to be the minimum expected total cost from node n at time t to γ through the link (m, m') when all observed links are

deterministically uncongested, and $\underline{v}(n, (m, m'))$ to be the minimum total cost from node n to γ through the link (m, m') when $\underline{c}(n, n')$ is assigned to each observed link $(n, n') \in A$ for all $t \leq T$. Thus, by using the definition of $\underline{c}(n, n')$ and $\underline{c}(n, n')$ we have

$$\bar{u}(n, t) \leq \overline{u(n)}, \quad (3)$$

$$\underline{v}(n) \leq \underline{v}(n, t), \quad (4)$$

$$\underline{v}(n, (m, m')) \leq \underline{v}(n, t, (m, m')). \quad (5)$$

Suppose we have

$$\bar{u}(n, t) < \underline{v}(n, t, (m, m')). \quad (6)$$

That is, the expected total cost in the worst traffic condition from node n at time t to γ along the predetermined path is less than the one in the best traffic condition through the link (m, m') . Upon noting that $v^*(n, t, z) \leq \bar{u}(n, t)$, it is immediate when (6) holds that we need not consider the congestion status of the link (m, m') at time t regardless of any possible vector z .

Suppose n' is the immediate successor node of n along $F(n, \gamma)$. If (6) implies

$$\bar{u}(n', t') < \underline{v}(n', t', (m, m')) \quad (7)$$

for all $t' > t$, then by noting that $v^*(n', t', z') \leq \bar{u}(n', t')$, the congestion status of the link (m, m') that was deleted from consideration at node n remains unimportant in the subsequent steps. We can then solve the optimality equations with the reduced state space by eliminating the congestion status of the link (m, m') from consideration without loss of optimality.

Since the link costs are non-stationary, we note that computing the quantities $\bar{u}(n, t)$ and $\underline{v}(n, t, (m, m'))$ for all t may require considerable computation. Thus, we introduce a more efficient procedure that can be performed on a stationary, deterministic road network. The remainder of this section is devoted to proving the following result.

Theorem 1: Given $F(n, \gamma)$, suppose for an observed link (m, m') that

$$\overline{u(n)} < \underline{v}(n, (m, m')). \quad (8)$$

Then the congestion status of the link (m, m') need not be considered when computing an optimal policy from n to γ for any time t .

This inequality states that the upper-bound on the total travel cost in the worst traffic condition from node n to γ along the predetermined path is less than the lower bound in the best traffic condition through the link (m, m') . If (8) holds, then equations (3) and (5) imply

$$\bar{u}(n, t) < \underline{v}(n, t, (m, m')) \quad (9)$$

and the previous argument yields that (m, m') does not effect the optimal choice at node n . In order to complete the proof of Theorem 1 we must show that we need not observe the congestion status of (m, m') as the trip progresses. To show that (6) implies (7), we require the following proposition.

Proposition 2: Given $F(n, \gamma)$, suppose a link (m, m') is observed, and

$$\overline{u(n)} < \underline{v}(n, (m, m')), \quad (10)$$

i.e., we need not observe (m, m') at time t . Then for n' , the immediate successor node of n along $F(n, \gamma)$,

$$\overline{u(n')} < \underline{v}(n', (m, m')). \quad (11)$$

Proof: It should be clear that $F(n, \gamma)$ may be sub-optimal for the original (non-stationary, stochastic) network. Define $k(a, b)$ to be the minimum total cost from a to b when $\underline{c}(n, n'')$ is assigned to each observed link $(n, n'') \in A$. Suppose (10) holds. Then, by definition

$$\overline{u(n)} < \underline{v}(n, (m, m')) = \underline{k}(n, m) + \underline{c}(m, m') + \underline{k}(m', \gamma). \quad (12)$$

Note that $\underline{v}(n, (m, m'))$ is obtained from a road network with stationary, deterministic link costs.

Suppose n' is the immediate successor node of n along the predetermined path. Consider the case when the link (n, n') is observed. Then,

$$\begin{aligned} \underline{k}(n, m) - \underline{k}(n', m) &\leq \underline{c}(n, n') \\ &\leq \underline{c}(n, n') = \overline{u(n)} - \overline{u(n')}, \end{aligned} \quad (13)$$

where the first inequality is an equality if the link (n, n') having $\underline{c}(n, n')$ as the link cost is on the minimum cost route from node n to m . The equality at the end of (13) follows from the fact that n' is the immediate successor node of n along the predetermined path.

Now, consider the case when the link (n, n') is unobserved, and the deterministic link cost $\underline{c}(n, n') \equiv \underline{c}(n, n') = \underline{c}(n, n')$ is assigned to the link (n, n') . We have

$$\underline{k}(n, m) - \underline{k}(n', m) \leq \underline{c}(n, n') = \overline{u(n)} - \overline{u(n')}, \quad (14)$$

where again, if the link (n, n') is on the minimum cost route from node n to m , the inequality is an equality. Combining (13) and (14) we get

$$\underline{k}(n, m) - \underline{k}(n', m) \leq \overline{u(n)} - \overline{u(n')}. \quad (15)$$

By adding (12) and (15), we obtain

$$\overline{u(n')} < \underline{k}(n', m) + \underline{c}(m, m') + \underline{k}(m', \gamma),$$

which implies (11), and the proof is complete. \blacksquare

The next proposition is an immediate consequence of the inequalities (3), (5) and Proposition 2.

Proposition 3: Given $F(n, \gamma)$, suppose a link (m, m') is observed, and

$$\overline{u(n)} < \underline{v}(n, (m, m')). \quad (16)$$

Then

$$\bar{u}(n, t) < \underline{v}(n, t, (m, m')) \quad (17)$$

implies

$$\bar{u}(n', t') < \underline{v}(n', t', (m, m')). \quad (18)$$

We are now ready to prove Theorem 1.

Proof of Theorem 1:

Proof: Applying (8) implies

$$v^*(n, t, z) < \underline{k}(n, m) + \underline{c}(m, m') + \underline{k}(m', \gamma), \quad (19)$$

by noting that $v^*(n, t, z) \leq \overline{u(n)}$. Let n' be the immediate successor node of n determined by an optimal policy. By using a similar argument to that in Proposition 2

$$v^*(n', t', z') < \underline{k(n', m)} + \underline{c(m, m')} + \underline{k(m', \gamma)}, \quad (20)$$

for any t' , and the proof is complete. ■

While Theorem 1 holds for any node in the network, we are concerned in particular with the start node, n_0 . Table I summarizes a procedure for eliminating unnecessary observations before the trip begins. Again refer to Figure 1. In practice, when this reduction is complete, the optimality equations have become more tractable and we can inductively solve them and create a look-up table of an optimal policy. In the next section, we discuss how to further eliminate unnecessary observations and reduce the portion of the table that must be accessed dynamically as the trip progresses.

B. Dynamic state space reduction

In this section we show that we may further reduce the state space as the trip progresses thereby eliminating unnecessary observations and reducing the need to access the whole table. To make this more concrete, consider the situation depicted in Figure 2. Some observed links (the short lines in the figure) may be important when the trip begins, but may no longer be important as the trip progresses toward the goal node. Of course, the same procedure that is outlined in the previous section could be used to prune the network as the vehicle reaches each node by redefining the origin, but this may not be desirable since it would require re-solving the problem several times. We seek a much simpler approach.

We begin by showing that the optimality equations yield conditions for state space reduction as the trip optimally progresses. The first two conditions, A1-2 below, can be verified after the a priori state space reduction has been completed since v^* is available.

Suppose z is partitioned so that $z = (\tilde{z}, \hat{z})$ where \tilde{z} is a $(Q - 1)$ -dimensional vector describing the transitions of $Q - 1$ observed links and $\hat{z} \in \{0, 1\}$ describes the remaining observed link. Consider the following assumption :

A1: For all (n, t, \tilde{z}) and $n' \in SCS(n)$

$$h(n, t, (\tilde{z}, \hat{z} = 0), n', v^*) \leq h(n, t, (\tilde{z}, \hat{z} = 1), n', v^*).$$

Proposition 4: The following statements hold

1) Suppose Assumption A1 holds. Then

$$v^*(n, t, (\tilde{z}, \hat{z} = 0)) \leq v^*(n, t, (\tilde{z}, \hat{z} = 1)).$$

2) Suppose for given n, t and all \tilde{z}

$$v^*(n, t, (\tilde{z}, \hat{z} = 0)) = v^*(n, t, (\tilde{z}, \hat{z} = 1)). \quad (21)$$

Then,

$$\begin{aligned} & \arg \min_{n' \in SCS(n)} \{h(n, t, (\tilde{z}, \hat{z} = 0), n', v^*)\} \\ & \bigcap_{n' \in SCS(n)} \arg \min_{n' \in SCS(n)} \{h(n, t, (\tilde{z}, \hat{z} = 1), n', v^*)\} \neq \emptyset. \quad (22) \end{aligned}$$

Proof: It follows directly from the optimality equations that A1 implies

$$v^*(n, t, (\tilde{z}, \hat{z} = 0)) \leq v^*(n, t, (\tilde{z}, \hat{z} = 1));$$

the first assertion is trivial. The second assertion is proved by contradiction. Suppose (21) holds and assume

$$\begin{aligned} & \arg \min_{n' \in SCS(n)} \{h(n, t, (\tilde{z}, \hat{z} = 0), n', v^*)\} \\ & \bigcap_{n' \in SCS(n)} \arg \min_{n' \in SCS(n)} \{h(n, t, (\tilde{z}, \hat{z} = 1), n', v^*)\} = \emptyset. \end{aligned}$$

Then, for

$$n^0 \in \arg \min_{n' \in SCS(n)} \{h(n, t, (\tilde{z}, \hat{z} = 0), n', v^*)\}$$

and

$$n^1 \in \arg \min_{n' \in SCS(n)} \{h(n, t, (\tilde{z}, \hat{z} = 1), n', v^*)\},$$

we have

$$\begin{aligned} v^*(n, t, (\tilde{z}, \hat{z} = 0)) &= h(n, t, (\tilde{z}, \hat{z} = 0), n^0, v^*) \\ &< h(n, t, (\tilde{z}, \hat{z} = 0), n^1, v^*) \\ &\leq h(n, t, (\tilde{z}, \hat{z} = 1), n^1, v^*) \\ &= v^*(n, t, (\tilde{z}, \hat{z} = 1)), \end{aligned}$$

where the second inequality follows from A1. This contradicts (21), and the result follows. ■

Proposition 4 implies that if (21) holds, we cannot improve the optimal expected total cost by observing \hat{z} at node n at time t . That is to say, under (21), (in state $(n, t, (\tilde{z}, \hat{z}))$) there exists a choice of node that results in the same expected cost, regardless of the congestion status of the link corresponding to \hat{z} . We conclude that the observed link corresponding to \hat{z} provides superfluous information for given n, t and all \tilde{z} , and we can eliminate \hat{z} from the state space without loss of optimality.

Consider the following assumptions :

A2: For all (n, t, \tilde{z}) , if (n, n') is the observed link corresponding to \hat{z} , then $h(n, t, (\tilde{z}, \hat{z} = 0), n', v^*) < h(n, t, (\tilde{z}, \hat{z} = 1), n', v^*)$.

A3: For all (n, t, \tilde{z}) , if (n, n') is not the observed link corresponding to \hat{z} , then $c(n, t, (\tilde{z}, \hat{z} = 0), n') = c(n, t, (\tilde{z}, \hat{z} = 1), n')$, and $P(t'|n, t, (\tilde{z}, \hat{z} = 0), n') = P(t'|n, t, (\tilde{z}, \hat{z} = 1), n')$.

A4: For all (n, t, \tilde{z}) , $n' \in SCS(n)$, every $t' > t$ and \tilde{z}' such that $P(t'|n, t, (\tilde{z}, \hat{z}), n')P(\tilde{z}'|t, \tilde{z}, t') > 0$ for all \hat{z} ,

$$P(\hat{z}' = 1|t, \hat{z} = 0, t') < P(\hat{z}' = 1|t, \hat{z} = 1, t').$$

Theorem 5 and a simple induction argument state that once the observation of \hat{z} is identified as redundant by Proposition 4, it remains redundant as the trip optimally progresses.

Theorem 5: Suppose Assumptions A1-A4 hold and for given n, t and all \tilde{z}

$$v^*(n, t, (\tilde{z}, \hat{z} = 0)) = v^*(n, t, (\tilde{z}, \hat{z} = 1)). \quad (23)$$

Choose π^* so that $n^* = \pi^*(n, t, (\tilde{z}, \hat{z} = 0)) = \pi^*(n, t, (\tilde{z}, \hat{z} = 1))$. Then, for every $t' > t$ and \tilde{z}' such that $P(t'|n, t, (\tilde{z}, \hat{z}), n^*)P(\tilde{z}'|t, \tilde{z}, t') > 0$ for all \tilde{z} ,

$$v^*(n^*, t', (\tilde{z}', \hat{z}' = 0)) = v^*(n^*, t', (\tilde{z}', \hat{z}' = 1)) .$$

Proof: Suppose (23) holds. It follows from Proposition 4 that π^* can be chosen so that $n^* = \pi^*(n, t, (\tilde{z}, \hat{z} = 0)) = \pi^*(n, t, (\tilde{z}, \hat{z} = 1))$.

Let (n, n^*) be the observed link corresponding to \hat{z} . Then, it follows from A2 that

$$\begin{aligned} v^*(n, t, (\tilde{z}, \hat{z} = 0)) &= h(n, t, (\tilde{z}, \hat{z} = 0), n^*, v^*) \\ &< h(n, t, (\tilde{z}, \hat{z} = 1), n^*, v^*) \\ &= v^*(n, t, (\tilde{z}, \hat{z} = 1)) , \end{aligned}$$

which is a contradiction. Thus, we conclude (n, n^*) is not the observed link corresponding to \hat{z} .

Straightforward algebraic manipulation and A3 imply that

$$\begin{aligned} 0 &= v^*(n, t, (\tilde{z}, \hat{z} = 1)) - v^*(n, t, (\tilde{z}, \hat{z} = 0)) \\ &= c(n, t, (\tilde{z}, \hat{z} = 1), n^*) - c(n, t, (\tilde{z}, \hat{z} = 0), n^*) \\ &\quad + \sum_{t'} P(t'|n, t, (\tilde{z}, \hat{z} = 1), n^*) \sum_{\tilde{z}'} P(\tilde{z}'|t, \tilde{z}, t') \\ &\quad \times \sum_{\tilde{z}'} P(\tilde{z}'|t, \hat{z} = 1, t') v^*(n^*, t', (\tilde{z}', \hat{z}')) \\ &\quad - \sum_{t'} P(t'|n, t, (\tilde{z}, \hat{z} = 0), n^*) \sum_{\tilde{z}'} P(\tilde{z}'|t, \tilde{z}, t') \\ &\quad \times \sum_{\tilde{z}'} P(\tilde{z}'|t, \hat{z} = 0, t') v^*(n^*, t', (\tilde{z}', \hat{z}')) \\ &= \sum_{t'} P(t'|n, t, (\tilde{z}, \hat{z} = 1), n^*) \sum_{\tilde{z}'} P(\tilde{z}'|t, \tilde{z}, t') \\ &\quad \times \left(P(\tilde{z}' = 1|t, \hat{z} = 1, t') - P(\tilde{z}' = 1|t, \hat{z} = 0, t') \right) \\ &\quad \times \left(v^*(n^*, t', (\tilde{z}', \hat{z}' = 1)) - v^*(n^*, t', (\tilde{z}', \hat{z}' = 0)) \right), \end{aligned}$$

where we note that

$$\begin{aligned} P(\tilde{z}' = 0|t, \hat{z} = 0, t') &= 1 - P(\tilde{z}' = 1|t, \hat{z} = 0, t'), \\ P(\tilde{z}' = 0|t, \hat{z} = 1, t') &= 1 - P(\tilde{z}' = 1|t, \hat{z} = 1, t') . \end{aligned}$$

The result then follows from A1 and A4. \blacksquare

We note that all of the results of this section thus far are heavily dependent on A1 and A2. As previously noted, these can be verified after the a priori state space reduction has been performed. On the other hand, there are problem instances in which these conditions can be shown to hold even before the a priori state space reduction. We now present conditions that imply that these two key assumptions hold. Consider the following assumptions :

- A5:** At every node that connects an observed link, the driver can be made to wait; that is, $n \in SCS(n)$.
- A6:** The cost rate of waiting is less than that of driving to the node; $c(n, t, z, n, t + 1) < c(n', t, z, n, t + 1)$ for all $n \neq n'$.
- A7:** The time to traverse any link when it is uncongested is stochastically less than when it is congested. That is, if the time to traverse link (n, n') at time t is

denoted $T^c(n, t, n')$ or $T^u(n, t, n')$ when the link is congested or uncongested, respectively, then

$$P(T^u(n, t, n') \geq k) \leq P(T^c(n, t, n') \geq k) \quad (24)$$

for all k . We further assume that (24) has strict inequality for at least one k .

- A8:** If $z_k \leq \bar{z}_k$, then the process that starts at z_k , say $Z \equiv \{z(t_i) = z_i, i \geq k\}$, is stochastically less than that which starts at \bar{z}_k , $\bar{Z} \equiv \{\bar{z}(t_i) = \bar{z}_i, i \geq k\}$ (written $Z \leq_{st} \bar{Z}$).

Assumptions A7 and A8 require some additional remarks. Assumption A7 is quite reasonable given the fact that in most applications, the ordering is with probability 1; a congested link takes more time to travel than an uncongested one. Stochastic ordering is a significantly weaker assumption. Furthermore, a sufficient condition for the stochastic ordering in Assumption A8 to hold for each q is that $1 - \alpha_t^q \leq \beta_t^q$ for all t . This implies the stochastic ordering of each individual Markov chain which in turn implies the stochastic ordering of the processes Z and \bar{Z} (cf. Kulkarni [24].). The main result on stochastic ordering of stochastic processes that we will use is called the *coupling* theorem. We state this theorem for completeness.

Theorem 6: (cf. Kulkarni [24].) Consider stochastic processes $X = \{X_n, n \geq 0\}$ and $Y = \{Y_n, n \geq 0\}$. $X \leq_{st} Y$ if and only if, there exist two stochastic processes $\hat{X} = \{\hat{X}_n, n \geq 0\}$ and $\hat{Y} = \{\hat{Y}_n, n \geq 0\}$ on a common probability space such that

- 1) $X = \hat{X}$ and $Y = \hat{Y}$ in distribution
- 2) $\hat{X}_n \leq \hat{Y}_n$ for all $n \geq 0$ almost surely (with probability one).

The next result shows that under Assumptions A5-A8 the required inequalities on h hold.

Theorem 7: Suppose Assumptions A5-A8 hold and, $z \leq \bar{z}$. Then, A1 and A2 hold.

Proof: The result is proved via a sample path argument. Since $Z \leq_{st} \bar{Z}$ we may construct a single probability space such that $Y \equiv \{y(t_i), i \geq 0\} = Z$ and $\bar{Y} \equiv \{\bar{y}(t_i), i \geq 0\} = \bar{Z}$ in distribution and $y(t_i) \leq \bar{y}(t_i)$ with probability one for all i , where $t_0 = t$. Similarly, the times to traverse a link (n_i, n_{i+1}) that is uncongested for Y but congested for \bar{Y} , say $T^u(n_i, t_i, n_{i+1})$ and $\bar{T}^c(n_i, t_i, n_{i+1})$, are constructed such that $T^u(n_i, t_i, n_{i+1}) \leq \bar{T}^c(n_i, t_i, n_{i+1})$ almost surely but are equal in distribution to those that would be constructed for the original processes Z and \bar{Z} . To complete the construction, when the two processes are the same on a particular link, we use a common random variable to construct the travel time.

Suppose we start two vehicles at time t from node n with the goal of reaching γ . The first vehicle sees congestion status z initially, while the second sees congestion status \bar{z} . The second vehicle follows the policy, π' that moves to n' from its current state and then follows an optimal policy. The first uses a policy, π , that in essence follows the same policy as π' , but if it arrives to node n' earlier than vehicle 2, π requires vehicle 1 to wait for vehicle 2. For example, suppose (n, n') is observed. Since $y(t) = z \leq \bar{y}(t) = \bar{z}$ at time t , if (n, n') is congested for vehicle 1, it is also congested for vehicle 2 and the travel

times are the same. Similarly, if (n, n') is uncongested for vehicle 2, it is also uncongested for vehicle 1. On the other hand, if the link is uncongested for vehicle 1, and congested for vehicle 2, we are in the scenario described in A2 (we are comparing $\hat{z} = 0$ to $\hat{z} = 1$). Our assumption that there is strict inequality for some k in the stochastic ordering relation implies that on some sample paths (with positive probability) the time to traverse the link for vehicle 1 is (strictly) less than that of vehicle 2. Assume we are on one of those paths. Upon arrival to node n' at time t' say, vehicle 1 waits for vehicle 2 to arrive and is charged $c(n', t', z', n', t' + 1)$ for each time unit until vehicle 2 arrives. Suppose vehicle 2 arrives at node n' at time t'' . Vehicle 2 has accrued travelling costs from t to t'' while vehicle 1 has accrued travelling costs from t to t' and waiting costs from t' to t'' . Thus, by assumption, the cost of vehicle 2 is (strictly) greater than that of vehicle 1. Continuing in this manner for the entire path of vehicle 2 we have, the total expected cost to be accrued for each vehicle is

$$\begin{aligned} C(n, t, z) &\equiv \sum_{i=0}^M c(n_i, t_i, y_i, \pi(n_i, t_i, y_i)) + \bar{c}(n_{M+1}, t_{M+1}) \\ &< \sum_{i=0}^M c(n_i, t_i, \bar{y}_i, \pi'(n_i, t_i, \bar{y}_i)) + \bar{c}(n_{M+1}, t_{M+1}) \\ &\equiv C(n, t, \bar{z}). \end{aligned}$$

Taking expectations yields $\mathbb{E}(C(n, t, z)) < \mathbb{E}(C(n, t, \bar{z}))$. Since the Z (\bar{Z}) and Y (\bar{Y}) are equal in distribution we have

$$\begin{aligned} h(n, t, (\bar{z}, \hat{z} = 0), n', v^*) &\leq \mathbb{E}(C(n, t, (\bar{z}, \hat{z} = 0))) \\ &< \mathbb{E}(C(n, t, (\bar{z}, \hat{z} = 1))) = h(n, t, (\bar{z}, \hat{z} = 1), n', v^*), \end{aligned}$$

and A2 is proven. The (non-strict) inequality described in A1 is due to the cases where the travel times are almost surely equivalent; either the link (n, n') is unobserved or in the same state for both processes. ■

In the next section we show how the main results of this section can be applied to a real road network with actual traffic data. In particular, we show that we may reduce the state space before the trip begins and then continue to reduce the state space as the trip progresses, making a difficult problem computationally manageable.

V. NUMERICAL RESULTS

We consider the road network in southeast Michigan depicted in Figure 3. The network consists of 139 nodes and 213 arcs. Some of the arcs corresponding to highways are fitted with traffic sensors to provide real-time traffic data. In this network, there are 49 such observed arcs. In 2000, the current work was funded by the Michigan Department of Transportation to decide if the cost savings obtained from optimal use of these data would warrant further investment in such equipment throughout the state. Thirty days of actual one minute traffic data were provided for all of the observed links. It was suggested (and confirmed) that a reasonable assumption for the unobserved links was to deterministically set the traffic speed to the speed limit on the link in question. Since the state space of the problem was so large (recall the 2^Q components of

the total number of states in the Markov chain for the observed link), the current study was undertaken so as to make the problem more tractable. The actual cost savings of having the network equipped with traffic sensors is reported in Kim et al. [20].

To show the usefulness of our methods we investigate 30 instances by randomly choosing origin/destination pairs and the departure time. We use a Pentium 4 personal computer with 3.06GHz CPU and 2G RAM. First, we examine how the reduced number of observed links by the a priori state space reduction can affect the total states expanded and actions evaluated. This results in a substantial decrease in add/multiply operations and, consequently, CPU time. Table II summarizes the results of our study. It implies that the reduced number of observed links directly affects the computation time. CPU time decreases to, on average, only 2.55% of that of the original problem after applying the reduction.

Figure 4 shows the histogram for the CPU time ratios of 'after' to 'before' a priori state space reduction. We note that the ratios are actually less than 0.01 in many cases and less than 0.05 in most instances, which implies our introduced methods are robust and result in significant computational improvement in diverse transportation environments.

Figure 5 shows how to further reduce the required observations and decrease the portion of the table that must be accessed dynamically as the trip optimally progresses. For example, as we progress 30% of the total trip, we do not need, on average, over 80% of the original observations, and as we progress 50% of the trip, approximately 90% of the original observations can be eliminated. Considering the fast increasing number of real-time traffic observations in large urban areas, we note that these percent reductions can be significant.

VI. CONCLUSIONS

This paper presents a procedure of state space reduction for non-stationary stochastic shortest path problems with real-time traffic congestion information. Our main conclusion is that this method can significantly improve computational tractability by systematically reducing the state space, without loss of optimality. This procedure exploits the fact that fast deterministic search algorithms (such as A^*) can aid in reducing the computational requirements of non-stationary stochastic shortest path problems.

Although we have presented methods for reducing the state space of a problem with the congestion status of each link being modelled as a two-state Markov chain, the methods are extendable to the case when the congestion status of each link can be in one of $2 \leq W < \infty$ states. The definitions of congestion status are likely to be problem specific based on the typical travel times along each link. Moreover, we have made the assumption that the link travel times are independent. At present, it is not clear if our results can be extended to the case where this does not hold. In particular, comparisons made between the cost and probability functions such as those A3 may have to be adjusted.

In the a priori state space reduction process, we have left the choice of the (potentially sub-optimal) predetermined path

up to the decision-maker. It is often the case that a driver has made the trip from the source to the destination regularly and has a preferred path that can be used in this step. As an alternative, if the preferred path does not yield much in the way of state space reduction, one might choose several paths to see which yields the most benefit. If one prefers a more systematic approach, the shortest path in the deterministic network with the cost function being $c(n, n')$ on each link is also a candidate. The key observation is that when better bounds are available, the state space reduction algorithm will delete more unnecessary links, thereby decreasing the solution time of the optimality equations. This of course is true in the other direction as well; if the bounds are not tight, less observations can be ignored.

We also mention that we may be able to apply the a priori state space reduction process backward from the goal node set to the origin iteratively to obtain tighter bounds before solving the optimality equations. Investigating trade-offs between more computations and tighter bounds is an interesting future research topic. Finally, we remark that the assumption that the unobserved links have deterministic cost functions is only a restriction in the sense that we require that the cost functions are stationary. If the cost is to be a random amount, we would then use the expected cost along each link as the deterministic cost function.

APPENDIX I

LIST OF RELEVANT NOTATION

- $G = (N, A)$: The underlying road network
- N : The set of nodes
- A : The set of arcs
- Γ : The goal node set
- $SCS(n)$: The successor set of a node n
- Q : The number of observed links
- $z(t) = \{z^1(t), \dots, z^Q(t)\}$: The road congestion status vector
- α_t^q (β_t^q): Given the q^{th} link is uncongested (congested) at time t , the probability that it is uncongested (congested) at time $t + 1$
- $P(t'|n, t, z, n')$: the probability of arriving at node n' at time t' , given that the vehicle travels from node n to n' , departing node n at time t with congestion status vector z
- n_k : The k^{th} node visited by a vehicle
- t_k : The time n_k is visited
- T : The time after which no decisions can be made
- $\bar{c}(n, t)$: Terminal cost of completing a trip
- L : Terminal cost of not completing a trip
- U : The set of possible decision epochs
- Ω : The state space
- π : A generic element of the set of deterministic, Markovian policies
- $\bar{c}(n, t, z, n', t')$: The cost accrued by traversing road segment (n, n') , given that travel begins at time t and ends at time t' with the congestion status z at time t
- $c(n, t, z, n')$: The expected cost of going from n to n' starting at time t given congestion status z

- b (B): Lower (upper) bound on c
- \hat{T} : Time before which we are guaranteed to have path from any origin to the goal node set
- v^π : The total expected cost accrued under policy π
- $\mathbb{E}_{n_0, t_0, z_0}^\pi$: Expectation operator under policy π conditioned on the initial state (n_0, t_0, z_0)
- v^* : Optimal total expected cost
- $h(n, t, z, n', f)$: The total expected cost when in state (n, t, z) , n' is chosen as the next node to visit and a terminal cost f is accrued after moving to n'
- π^* : An optimal policy
- (\tilde{z}, \hat{z}) : A partitioning of the vector of observed links; \hat{z} represents the status of those links that will potentially be removed from the state space
- $F(n, \gamma)$: A predetermined path from n to γ (the goal node)
- $\underline{c}(n, t, n')$ ($\bar{c}(n, t, n')$): The expected link cost from n to n' when the observed link (n, n') is uncongested (congested) at time t
- $\underline{c}(n, n')$ ($\bar{c}(n, n')$): The minimum (maximum) of the above over all time
- $\bar{u}(n, t)$ ($u(n)$): The total cost along $F(n, \gamma)$ starting at time t when links are deterministically congested (assigned $(\bar{c}(n, n'))$)
- $\underline{v}(n, t)$ ($v(n)$): The minimum total cost from node n to γ when links are deterministically uncongested (assigned $(\underline{c}(n, n'))$)
- $\underline{v}(n, t, (m, m'))$ ($v(n, (m, m'))$): The minimum total cost from node n to γ at time t through the link (m, m') when all observed links are deterministically uncongested (assigned $(\underline{c}(n, n'))$)
- $\underline{k}(a, b)$: The minimum total cost from node a and b when all observed links are deterministically assigned $\underline{c}(n, n')$

REFERENCES

- [1] G. H. Polychronopoulos and J. N. Tsitsiklis, "Stochastic Shortest Path Problems with Recourse", *Networks*, vol. 27, pp. 133-143, 1996.
- [2] R. Hall, "The fastest path through a network with random time-dependent travel times", *Transportation Science*, vol. 20, pp. 182-188, 1986.
- [3] J. Pearl, *Heuristics : Intelligent search strategy for computer problem solving* Massachusetts: Addison-Wesley, 1984.
- [4] J. L. Bander and C. C. White III, "A Heuristic search algorithm for path determination with learning", *IEEE Transactions on Systems, Man, and Cybernetics - Part A: Systems and Humans*, vol. 28, pp. 131-134, 1998.
- [5] I. Chabini, "Adaptations of the A* algorithm for the computation of fastest paths in deterministic discrete-time dynamic networks", *IEEE Transactions on Intelligent Transportation Systems*, vol. 3, pp. 60-74, 2002.
- [6] J. L. Bander and C. C. White, III, "A Heuristic search approach for a nonstationary stochastic shortest path problem with terminal cost", *Transportation Science*, vol. 36, pp. 218-230, 2002.
- [7] G. R. Jagadeesh, T. Srikanthan and K. H. Quek, "Heuristic techniques for accelerating hierarchical routing on road networks", *IEEE Transactions on Intelligent Transportation Systems*, vol. 3, pp. 301-309, 2002.
- [8] E. D. Miller-Hooks and H. S. Mahmassani, "Least possible time paths in stochastic, time-varying transportation networks", *Computers and Operations Research*, vol. 25, pp. 1107-1125, 1998.
- [9] E. D. Miller-Hooks and H. S. Mahmassani, "Least expected time paths in stochastic, time-varying transportation networks", *Transportation Science*, vol. 34, pp. 198-215, 2000.
- [10] J. S. Croucher, "A Note on the Stochastic Shortest-Route Problem", *Naval Research Logistics Quarterly*, vol. 25, pp. 729-732, 1978.
- [11] G. Andreatta and L. Romeo, "Stochastic Shortest Paths with Recourse", *Networks*, vol. 18, pp. 193-204, 1988.

- [12] S. T. Waller and A. K. Ziliaskopoulos, "On the online shortest path problem with limited arc cost dependencies", *Networks*, vol. 40, pp. 216-227, 2002.
- [13] I. Chabini, "Discrete dynamic shortest path problems in transportation applications", *Transportation Research Record*, vol. 1645, pp. 170-175, 1998.
- [14] I. Chabini and S. Ganugapati, "Design and implementation of parallel dynamic shortest path algorithms for intelligent transportation systems applications", *Transportation Research Record*, vol. 1771, pp. 219-228, 2001.
- [15] S. Gao and I. Chabini, "Best routing policy problem in stochastic time-dependent networks", *Transportation Research Record*, vol. 1783, pp. 188-196, 2002.
- [16] H. N. Psaraftis and J. N. Tsitsiklis, "Dynamic shortest path in acyclic network with Markovian arc costs", *Operations Research*, vol. 41, pp. 91-101, 1993.
- [17] L. Fu and L. R. Rilett, "Expected Shortest Paths in Dynamic and Stochastic Traffic Networks", *Transportation Research Part B*, vol. 32, pp. 499-516, 1998.
- [18] L. Fu, "An adaptive routing algorithm for in vehicle route guidance systems with real-time information", *Transportation Research Part B*, vol. 35, pp. 749-765, 2001.
- [19] B. Thomas, "Anticipatory Route Selection Problems", Ph.D. thesis, University of Michigan, Ann Arbor, 2002.
- [20] S. Kim, M. E. Lewis and C. C. White III, "Optimal Vehicle Routing with Real-Time Traffic Information", To appear in the *IEEE Transactions on Intelligent Transportation Systems*, 2005.
- [21] S. M. Ross, *Stochastic processes* New York: Wiley, 1996.
- [22] M. L. Puterman, *Markov decision processes: Discrete stochastic dynamic programming* New York: Wiley, 1994.
- [23] S. Kim, "Optimal vehicle routing and scheduling with real-time traffic information", Ph.D. thesis, University of Michigan, Ann Arbor, 2003.
- [24] V. G. Kulkarni, *Modeling and analysis of stochastic systems* London: Chapman and Hall, 1995.



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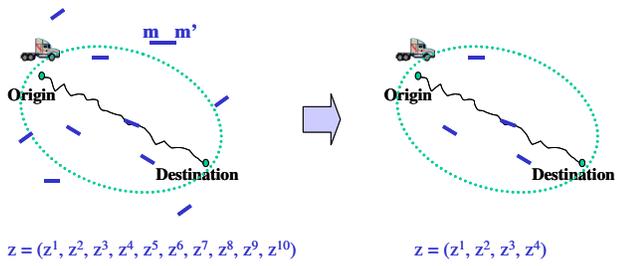


Fig. 1. An a priori state space reduction process. The short dashed lines represent observed links.

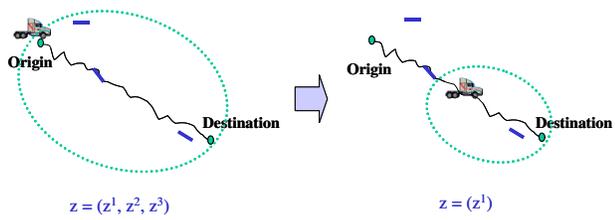


Fig. 2. A dynamic state space reduction process.

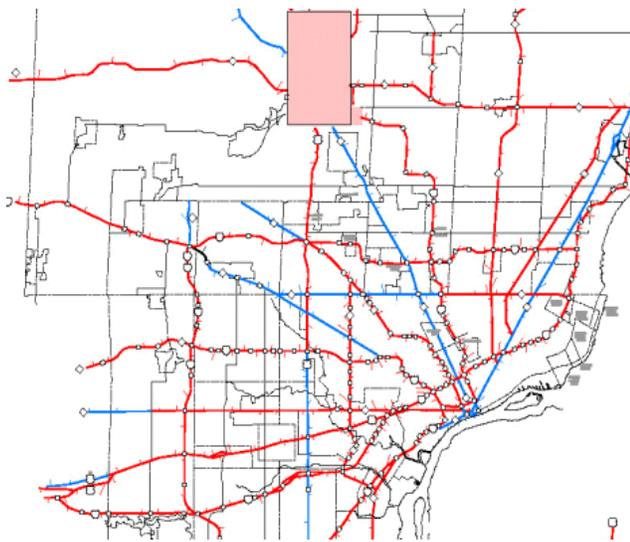


Fig. 3. A road network in southeast Michigan.

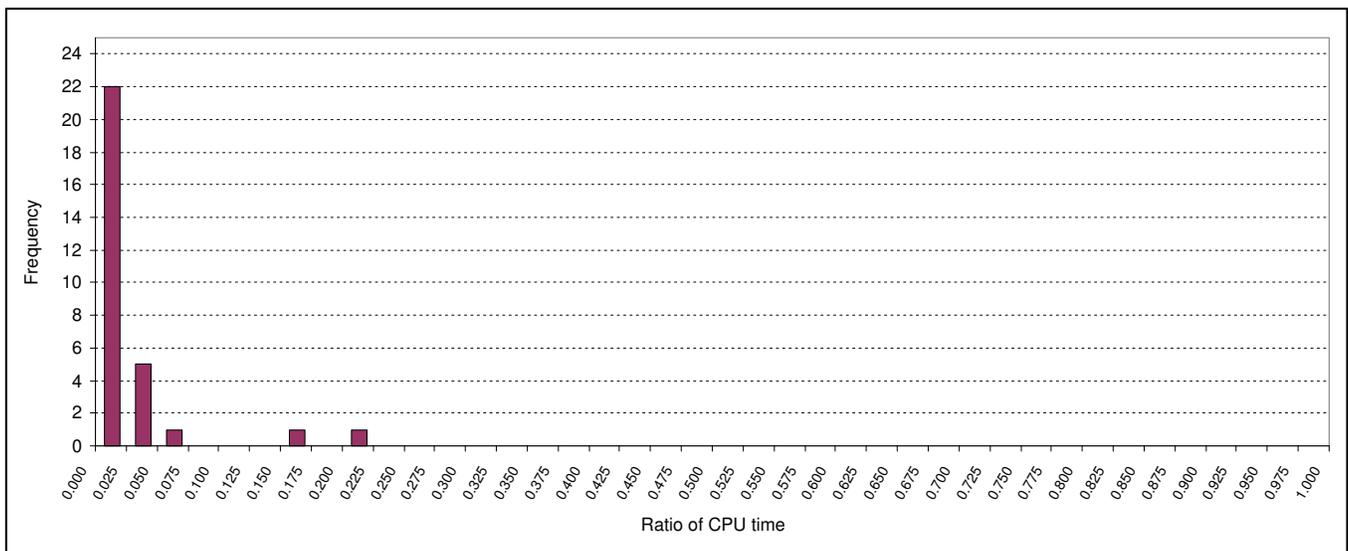


Fig. 4. Histogram of CPU ratio.

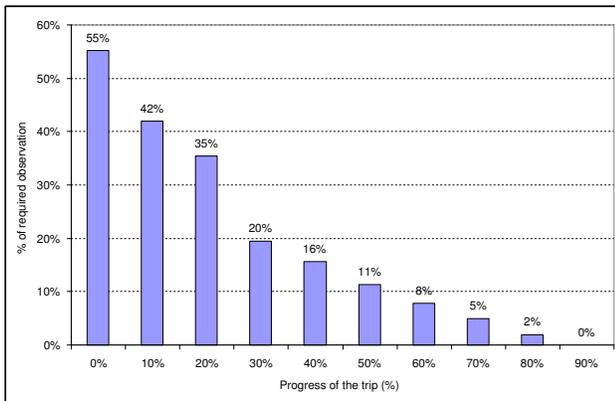


Fig. 5. Percent of required observations as the trip optimally progresses.

TABLE I

SUMMARY: A PROCEDURE FOR STATE SPACE REDUCTION BEFORE THE TRIP BEGINS.

- 1) Determine a “good” path from n_0 to γ , for example, using historical data or previous experience.
- 2) Calculate $\overline{u(n)}$ along the predetermined path by assigning $\overline{c(n, n')}$ for each observed link in the path.
- 3) Calculate $\overline{v(n, (m, m'))}$ through an observed link (m, m') by assigning $\overline{c(n, n')}$ for each observed link $(n, n') \in A$.
- 4) If $\overline{u(n)} < \overline{v(n, (m, m'))}$, then eliminate the congestion status of the link (m, m') from the state space z . Otherwise, go to 3 until all observed links have been evaluated.

TABLE II
AVERAGE RATIOS OF 'AFTER' TO 'BEFORE' A PRIORI STATE SPACE
REDUCTION.

	CPU Time	Add/Multiply Operations	States Expanded	Actions Evaluated
Average Ratio	0.0255	0.0383	0.1362	0.1366

CAPTIONS LIST

- Figure 1. An a priori state space reduction process. The short dashed lines represent observed links.
- Figure 2. A dynamic state space reduction process.
- Figure 3. A road network in southeast Michigan.
- Figure 4. Histogram of CPU ratio.
- Figure 5. Percent of required observations as the trip optimally progresses.
- Table I. Summary: A procedure for state space reduction before the trip begins.
- Table II. Average ratios of ‘after’ to ‘before’ a priori state space reduction.