A Refined Deterministic Linear Program for the Network Revenue Management Problem with Customer Choice Behavior

Sumit Kunnumkal,[1] Huseyin Topaloglu[2]

[1] *Indian School of Business, Gachibowli, Hyderabad, 500032, India*

[2] *School of Operations Research and Information Engineering, Cornell University, Ithaca, New York 14853*

Received 3 May 2007; revised 18 October 2007; accepted 23 March 2008
DOI 10.1002/nav.20296
Published online 21 July 2008 in Wiley InterScience (www.interscience.wiley.com).

Abstract: We present a new deterministic linear program for the network revenue management problem with customer choice behavior. The novel aspect of our linear program is that it naturally generates bid prices that depend on how much time is left until the time of departure. Similar to the earlier linear program used by van Ryzin and Liu (2004), the optimal objective value of our linear program provides an upper bound on the optimal total expected revenue over the planning horizon. In addition, the percent gap between the optimal objective value of our linear program and the optimal total expected revenue diminishes in an asymptotic regime where the leg capacities and the number of time periods in the planning horizon increase linearly with the same rate. Computational experiments indicate that when compared with the linear program that appears in the existing literature, our linear program can provide tighter upper bounds, and the control policies that are based on our linear program can obtain higher total expected revenues.

Keywords: revenue management; linear program; customer choice

1. INTRODUCTION

A prevalent assumption in the revenue management literature is that each customer arrives into the system with the intention of purchasing a particular itinerary. If its intended itinerary is available for purchase, then the customer purchases this itinerary. Otherwise, it does not purchase anything at all. In reality, however, there may be many different itineraries that are acceptable to a particular customer and the customer makes a choice among the acceptable itineraries that are available for purchase. This type of customer choice behavior is especially true nowadays with the Internet bringing a variety of itinerary choices to the customers.

Recently, [9] utilized a deterministic linear program that was first proposed by [4] to develop control policies for the network revenue management problem with customer choice behavior. This linear program includes one constraint for each flight leg and the right side of these constraints are the remaining leg capacities. Consequently, [9] used the optimal values of the dual variables associated with these capacity constraints to estimate the opportunity cost of a unit of capacity. They employ these opportunity costs to extend the popular bid pricing and dynamic programming decomposition ideas to the network revenue management problem with customer choice behavior.

In this article, we propose a new deterministic linear program for the network revenue management problem with customer choice behavior. Although one should intuitively expect the opportunity costs to decrease as the departure time of the flight legs approaches and fewer opportunities to utilize the leg capacities remain, the earlier linear program used by [9] essentially assumes that the opportunity costs of the leg capacities stay constant throughout the planning horizon. Our main objective in this article is to remedy this shortcoming. In particular, we propose a linear program that naturally generates opportunity costs that depend on the number of time periods left until the departure time. The hope is that our linear program captures the characteristics of the problem more accurately and obtains more refined opportunity costs.

The method that we use to construct our linear program is also of interest in and of itself. The linear program that appears in the existing literature is a deterministic and continuous approximation to the original problem. It is based on the *a priori* assumption that the random quantities take on their expected values and the itineraries can be sold in fractional amounts, in which case the network revenue management

Correspondence to: S. Kunnumkal (sumit_kunnumkal@isb.edu)

problem can be formulated as a linear program. The usual approach is to analyze how this linear program relates to the original problem through *a posteriori* analyses. On the other hand, we construct our linear program directly by using the dynamic programming formulation of the network revenue management problem. The fundamental idea is to relax the capacity availability constraints in the dynamic programming formulation by associating Lagrange multipliers with them, in which case the dynamic programming formulation decomposes by the time periods and we obtain simple expressions for the value functions. A good set of values for the Lagrange multipliers can be obtained by minimizing a dual function. The linear program that we propose in this paper essentially solves the problem of minimizing the dual function.

Our linear program shares the appealing features of the earlier linear program used by [9]. In particular, the optimal objective value of our linear program provides an upper bound on the optimal total expected revenue over the planning horizon. In an asymptotic regime where the leg capacities and the number of time periods in the planning horizon increase linearly with the same rate, the percent gap between the optimal objective value of our linear program and the optimal total expected revenue diminishes. Our linear program also allows us to extend the popular bid pricing and dynamic programming decomposition ideas to the network revenue management problem with customer choice behavior. On the other hand, when compared with the earlier linear program used by [9], computational experiments indicate that our linear program provides tighter upper bounds on the optimal total expected revenues and the performances of the control policies that are based on our linear program tend to be better. Furthermore, although we do not pursue here, it is straightforward to generalize our approach to incorporate cancellations by using the approach followed by [8]. This strengthens the links between the dynamic programming and linear programming formulations of the network revenue management problem.

Customer choice behavior is an active area of research. Belobaba and Weatherford [2] extend the expected marginal seat revenue heuristics of Belobaba [1] to incorporate the possibility that a customer buys a more expensive itinerary when the cheaper itinerary is closed. Talluri and van Ryzin [7] give a careful analysis of the single-leg revenue management problem with customer choice behavior and characterize the conditions under which protection level policies are optimal. Zhang and Cooper [12] consider parallel flights and provide decomposition methods to compute upper and lower bounds on the optimal total expected revenue over the planning horizon. Gallego et al. [4] analyze the benefits from selling flexible itineraries that allow the airlines to assign a customer to one of the alternative itineraries right before the departure time. The authors develop a linear program to approximate the optimal total expected revenue over the planning horizon. This linear program plays a crucial role in the network revenue management literature and it is subsequently used in [9] to develop control policies for the network revenue management problem with customer choice behavior. The particular focus of the latter paper is on using the linear program developed by Gallego et al. [4] to extend the bid pricing and dynamic programming decomposition ideas to deal with the customer choice behavior. Zhang and Adelman [11] developed the control policies by using the linear programming representation of the dynamic programming formulation of the network revenue management problem. Their approach is related to our linear program in the sense that it generates opportunity costs that depend on the number of time periods left until the departure time, but our linear program is considerably simpler. Finally, van Ryzin and Vulcano [10] compute protection levels by using a stochastic approximation method that avoids parametric assumptions about the model that governs the choice behavior of the customers.

We make the following research contributions in this article. (1) We present a new deterministic linear program for the network revenue management problem with customer choice behavior. The novel aspect of our linear program is that it naturally generates opportunity costs that depend on how much time is left until the time of departure. (2) We prove that the optimal objective value of our linear program provides an upper bound on the optimal total expected revenue over the planning horizon. In an asymptotic regime where the leg capacities and the number of time periods in the planning horizon increase linearly with the same rate, we establish that the percent gap between the optimal objective value of our linear program and the optimal total expected revenue diminishes. (3) The number of decision variables in our linear program increases exponentially with the number of itineraries, but we show that it is possible to solve our linear program efficiently by using standard column generation. (4) When compared with the deterministic linear program used by van Ryzin and Liu [9], computational experiments indicate that our linear program provides tighter upper bounds on the optimal total expected revenues and the performances of the control policies that are based on our linear program tend to be better.

The rest of the article is organized as follows. Section 2 formulates the problem as a dynamic program. Section 3 presents the earlier linear program used by van Ryzin and Liu [9]. Section 4 derives our linear program and shows that it provides an upper bound on the optimal total expected revenue. Section 5 compares the upper bounds provided by the two linear programs. This section also shows that the percent gap between the upper bound provided by our linear program and the optimal total expected revenue diminishes as the leg capacities and the number of time periods in the planning horizon increase linearly with the same rate. Section 6 describes different control policies that are based on the linear programs in Sections 3 and 4. Section 7 shows that our linear program can be solved efficiently as long as the customer choice behavior is governed by the multinomial logit model with disjoint consideration sets. Section 8 presents computational experiments.

2. PROBLEM FORMULATION

We have a set of flight legs to serve the customers that arrive over time with the intention of purchasing itineraries. At each time period, we need to decide which itineraries to offer to the customers. Each customer reviews the offered itineraries and purchases at most one of them according to a probability distribution defined over the set of offered itineraries. A sold itinerary generates a revenue and consumes the capacities on the relevant flight legs.

The set of flight legs in the airline network is $\mathcal{L}$ and the set of itineraries that can be offered to the customers is $\mathcal{J}$. The initial capacity on flight leg i is c_i. If a customer purchases itinerary j, then we generate a revenue of r_j and consume a_{ij} units of capacity on flight leg i. Naturally, we have $a_{ij} = 0$ when itinerary j does not include flight leg i. The problem takes place over the planning horizon $\mathcal{T} = \{1, \ldots, \tau\}$ and all flight legs depart at time period $\tau + 1$. We assume that the time periods correspond to small time intervals so that there is at most one customer arrival at each time period. The probability that there is a customer arrival at each time period is λ. If the set of itineraries that we offer to the customers is $\mathcal{S}$, then a customer purchases itinerary j with probability $P_j(\mathcal{S})$. Naturally, we have $P_j(\mathcal{S}) = 0$ when $j \notin \mathcal{S}$. We use $P_\phi(\mathcal{S}) = 1 - \sum_{j\in\mathcal{S}} P_j(\mathcal{S})$ to denote the probability that a customer does not purchase an itinerary. We assume that the arrivals in different time periods and the purchasing decisions of different customers are independent of each other. As evident from our notation, we also assume that the probability that there is a customer arrival and the probability that a customer purchases a particular itinerary do not depend on the time period. This assumption is only for notational brevity and it is straightforward to allow these probabilities to depend on the time period. The objective is to maximize the total expected revenue over the planning horizon.

Using x_{it} to denote the remaining capacity on flight leg i at time period t, $x_t = \{x_{it} : i \in \mathcal{L}\}$ captures the state of the system. As a function of the remaining leg capacities, we need to decide which itineraries to offer at each time period. Since it is feasible to offer an itinerary only if we have enough capacity on all of the flight legs that are included in this itinerary, the set of itineraries that we can offer at time period t is

$$\mathcal{O}(x_t) = \{\mathcal{S} \subset \mathcal{J} : \mathbf{1}(j \in \mathcal{S})\, a_{ij} \le x_{it} \quad \forall\, i \in \mathcal{L}, j \in \mathcal{J}\},$$

where $\mathbf{1}(\cdot)$ is the indicator function. In this case, the optimal policy can be found by computing the value functions through the optimality equation

$$\begin{aligned} V_t(x_t) &= \max_{\mathcal{S}\in\mathcal{O}(x_t)} \Bigg\{ \sum_{j\in\mathcal{J}} \lambda\, P_j(\mathcal{S}) \Big[r_j + V_{t+1}\Big(x_t - \sum_{i\in\mathcal{L}} a_{ij}\, e_i \Big) \Big] \\ &\qquad + [1 - \lambda + \lambda\, P_\phi(\mathcal{S})] V_{t+1}(x_t) \Bigg\} \\ &= \max_{\mathcal{S}\in\mathcal{O}(x_t)} \Bigg\{ \sum_{j\in\mathcal{J}} \lambda\, P_j(\mathcal{S}) \Big[r_j + V_{t+1}\Big(x_t - \sum_{i\in\mathcal{L}} a_{ij}\, e_i \Big) \\ &\qquad - V_{t+1}(x_t) \Big] \Bigg\} + V_{t+1}(x_t), \end{aligned} \quad (1)$$

where e_i is the $|\mathcal{L}|$-dimensional unit vector with a one in the element corresponding to $i \in \mathcal{L}$ and the second equality follows from the fact that $P_\phi(\mathcal{S}) = 1 - \sum_{j\in\mathcal{S}} P_j(\mathcal{S})$; see van Ryzin and Liu [9]. Throughout the rest of the paper, we assume that $\lambda = 1$ for notational brevity. We note that this is equivalent to letting $\tilde{P}_j(\mathcal{S}) = \lambda\, P_j(\mathcal{S})$ and $\tilde{P}_\phi(\mathcal{S}) = 1 - \lambda + \lambda\, P_\phi(\mathcal{S})$ and working with the probabilities $\{\tilde{P}_j(\mathcal{S}) : j \in \mathcal{S}, \mathcal{S} \subset \mathcal{J}\}$ and $\{\tilde{P}_\phi(\mathcal{S}) : \mathcal{S} \subset \mathcal{J}\}$.

In the optimality equation above, the number of possible values for the state variable x_t increases exponentially with the number of flight legs and the number of possible values for the decision variable $\mathcal{S}$ increases exponentially with the number of itineraries. Therefore, it is quite difficult to solve this optimality equation. In the next two sections, we describe approximate methods that can be used to decide which itineraries to offer to the customers at each time period.

3. DETERMINISTIC LINEAR PROGRAM

An alternative to solving the optimality equation in (1) is to employ a deterministic and continuous approximation to the problem. This approximation assumes that the random quantities take on their expected values and the itineraries can be sold in fractional amounts. As a result, we obtain the linear programming formulation used by van Ryzin and Liu [9].

To formulate the linear program, we let $h_t(\mathcal{S})$ be the frequency with which we offer set $\mathcal{S}$ at time period t. In this case, the expected revenue at time period t is

$$\sum_{\mathcal{S}\subset\mathcal{J}} \sum_{j\in\mathcal{S}} P_j(\mathcal{S})\, r_j\, h_t(\mathcal{S}) = \sum_{\mathcal{S}\subset\mathcal{J}} R(\mathcal{S})\, h_t(\mathcal{S}),$$

where $R(\mathcal{S}) = \sum_{j \in \mathcal{S}} P_j(\mathcal{S})\, r_j$ is the expected revenue when we offer set $\mathcal{S}$. Similarly, using $Q_i(\mathcal{S}) = \sum_{j \in \mathcal{S}} P_j(\mathcal{S})\, a_{ij}$ to denote the expected capacity consumption on flight leg i when we offer set $\mathcal{S}$, the expected capacity consumption on flight leg i at time period t is

$$\sum_{\mathcal{S} \subset \mathcal{J}} \sum_{j \in \mathcal{S}} P_j(\mathcal{S})\, a_{ij}\, h_t(\mathcal{S}) = \sum_{\mathcal{S} \subset \mathcal{J}} Q_i(\mathcal{S})\, h_t(\mathcal{S}).$$

Therefore, we can use the optimal objective value of the linear program

$$Z_{LP} = \max \sum_{t \in \mathcal{T}} \sum_{\mathcal{S} \subset \mathcal{J}} R(\mathcal{S})\, h_t(\mathcal{S}) \tag{2}$$

$$\text{subject to} \sum_{t \in \mathcal{T}} \sum_{\mathcal{S} \subset \mathcal{J}} Q_i(\mathcal{S})\, h_t(\mathcal{S}) \le c_i \quad \forall\, i \in \mathcal{L} \tag{3}$$

$$\sum_{\mathcal{S} \subset \mathcal{J}} h_t(\mathcal{S}) = 1 \qquad \forall\, t \in \mathcal{T} \tag{4}$$

$$h_t(\mathcal{S}) \ge 0 \qquad \forall\, \mathcal{S} \subset \mathcal{J},\ t \in \mathcal{T} \tag{5}$$

as an approximation to the optimal total expected revenue over the planning horizon; see [9]. The decision variables in problem (2)–(5) are $\{h_t(\mathcal{S}) : \mathcal{S} \subset \mathcal{J}, t \in \mathcal{T}\}$. The first set of constraints ensure that the total expected capacity consumptions over the planning horizon do not exceed the leg capacities. The second set of constraints ensure that the total frequency with which we offer the sets at each time period is equal to one. Since the empty set is a subset of $\mathcal{J}$, the second set of constraints allow not offering an itinerary with a certain frequency.

We emphasize that by using the approach followed by van Ryzin and Liu [9], it is possible to reduce the number of decision variables in problem (2)–(5) by a factor of $|\mathcal{T}|$, but the way we present this problem is more useful for the subsequent development in the article. In addition, problem (2)–(5) allows time dependent probabilities of the form $\{P_{jt}(\mathcal{S}) : j \in \mathcal{S}, \mathcal{S} \subset \mathcal{J}, t \in \mathcal{T}\}$ simply by using $R_t(\mathcal{S}) = \sum_{j \in \mathcal{S}} P_{jt}(\mathcal{S})\, r_j$ and $Q_{it}(\mathcal{S}) = \sum_{j \in \mathcal{S}} P_{jt}(\mathcal{S})\, a_{ij}$ instead of $R(\mathcal{S})$ and $Q_i(\mathcal{S})$.

The number of decision variables in problem (2)–(5) increases exponentially with the number of itineraries. However, the number of constraints is only $|\mathcal{L}| + |\mathcal{T}|$ and this suggests solving problem (2)–(5) by using column generation. In Section 7, we briefly revisit solving problem (2)–(5) by using column generation under a particular choice of the probabilities $\{P_j(\mathcal{S}) : j \in \mathcal{S}, \mathcal{S} \subset \mathcal{J}\}$.

There are two primary uses of problem (2)–(5). First, this problem can be used to decide which itineraries to offer. In particular, letting $\{\hat{\pi}_i : i \in \mathcal{L}\}$ be the optimal values of the dual variables associated with constraints (3), the idea is to use $\hat{\pi}_i$ as the estimate of the opportunity cost of a unit of capacity on flight leg i. If the set of itineraries that we offer is $\mathcal{S}$, then the expected revenue that we obtain is $\sum_{j \in \mathcal{S}} P_j(\mathcal{S})\, r_j$ and the total expected opportunity cost of the consumed capacities is $\sum_{j \in \mathcal{S}} \sum_{i \in \mathcal{L}} P_j(\mathcal{S})\, a_{ij}\, \hat{\pi}_i$. Therefore, it is sensible to offer the feasible set of itineraries that maximize the difference between the expected revenue and the total expected opportunity cost of the consumed capacities. In other words, we can solve the problem

$$\max_{\mathcal{S} \in \mathcal{O}(x_t)} \left\{ \sum_{j \in \mathcal{S}} P_j(\mathcal{S}) \left[r_j - \sum_{i \in \mathcal{L}} a_{ij}\, \hat{\pi}_i \right] \right\} \tag{6}$$

to decide which itineraries to offer at time period t. In revenue management language, these estimates of the opportunity costs are called bid prices. Letting $\tilde{V}_t(x_t) = \sum_{i \in \mathcal{L}} \hat{\pi}_i\, x_{it}$ for all $t \in \mathcal{T}$ and noting that $\tilde{V}_{t+1}(x_t) - \tilde{V}_{t+1}(x_t - \sum_{i \in \mathcal{L}} a_{ij}\, e_i) = \sum_{i \in \mathcal{L}} a_{ij}\, \hat{\pi}_i$, it is easy to see that solving problem (6) to decide which itineraries to offer is equivalent to approximating $V_{t+1}(x_t)$ on the right side of (1) by $\tilde{V}_{t+1}(x_t)$.

Second, Gallego et al. [4] show that the optimal objective value of problem (2)–(5) provides an upper bound on the optimal total expected revenue. In other words, letting $c = \{c_i : i \in \mathcal{L}\}$, we have $V_1(c) \le Z_{LP}$. This information can be useful when assessing the optimality gap of a suboptimal decision rule such as the one in (6).

The decision rule in (6) implicitly assumes that the opportunity costs of the leg capacities stay constant throughout the planning horizon. In reality, however, one should expect the opportunity costs to decrease as the departure time approaches and fewer opportunities to utilize the leg capacities remain. In practical implementations, as the departure time approaches, the time dependent nature of the opportunity costs is "mimicked" by resolving problem (2)–(5) with the remaining number of time periods in the planning horizon and the remaining leg capacities. In the next section, we develop an alternative linear program that naturally generates bid prices that depend on the number of time periods left until the departure time. The hope is that this linear program captures the characteristics of the problem more accurately and is able to obtain more refined bid prices.

4. AN ALTERNATIVE DETERMINISTIC LINEAR PROGRAM

In this section, we develop a new linear program that generates bid prices that depend on the number of time periods left until the departure time. Noting the constraints captured by the set $\mathcal{O}(x_t)$ in the optimality equation in (1), the fundamental idea is to relax these constraints by associating the Lagrange multipliers $\alpha = \{\alpha_{ijt} : i \in \mathcal{L}, j \in \mathcal{J}, t \in \mathcal{T}\}$ with them. In other words, this idea suggests solving the optimality equation

$$V_t^\alpha(x_t) = \max_{\mathcal{S} \subset \mathcal{J}} \left\{ \sum_{j \in \mathcal{J}} P_j(\mathcal{S}) \Big[r_j + V_{t+1}^\alpha \big(x_t - \textstyle\sum_{i \in \mathcal{L}} a_{ij}\, e_i \big) - V_{t+1}^\alpha(x_t) \Big] - \sum_{i \in \mathcal{L}} \sum_{j \in \mathcal{J}} \alpha_{ijt}\, \mathbf{1}(j \in \mathcal{S})\, a_{ij} \right\} + \sum_{i \in \mathcal{L}} \sum_{j \in \mathcal{J}} \alpha_{ijt}\, x_{it} + V_{t+1}^\alpha(x_t), \quad (7)$$

where the superscripts in the value functions emphasize that the solution to the optimality equation above depends on the Lagrange multipliers. The next proposition shows that we obtain upper bounds on the value functions by solving the optimality equation in (7).

PROPOSITION 1: If the Lagrange multipliers are positive, then we have $V_t(x_t) \le V_t^\alpha(x_t)$.

PROOF: We show the result by induction over the time periods. It is easy to show the result for the last time period. Assuming that the result holds for time period $t+1$ and letting $\hat{\mathcal{S}}$ be an optimal solution to problem (1), we have

$$\begin{aligned} V_t^\alpha(x_t) \ge & \sum_{j \in \mathcal{J}} P_j(\hat{\mathcal{S}}) \left[r_j + V_{t+1}^\alpha \left(x_t - \sum_{i \in \mathcal{L}} a_{ij}\, e_i \right) \right] + \left[1 - \sum_{j \in \mathcal{J}} P_j(\hat{\mathcal{S}}) \right] V_{t+1}^\alpha(x_t) \\ & - \sum_{i \in \mathcal{L}} \sum_{j \in \mathcal{J}} \alpha_{ijt} \mathbf{1}(j \in \hat{\mathcal{S}})\, a_{ij} + \sum_{i \in \mathcal{L}} \sum_{j \in \mathcal{J}} \alpha_{ijt}\, x_{it} \\ \ge & \sum_{j \in \mathcal{J}} P_j(\hat{\mathcal{S}}) \left[r_j + V_{t+1} \left(x_t - \sum_{i \in \mathcal{L}} a_{ij}\, e_i \right) \right] + \left[1 - \sum_{j \in \mathcal{J}} P_j(\hat{\mathcal{S}}) \right] V_{t+1}(x_t), \end{aligned}$$

where the first inequality follows from the fact that $\hat{\mathcal{S}}$ is a feasible but not necessarily an optimal solution to problem (7) and the second inequality follows from the induction assumption and the fact that $\hat{\mathcal{S}} \in \mathcal{O}(x_t)$ and $\alpha_{ijt} \ge 0$ for all $i \in \mathcal{L}$, $j \in \mathcal{J}$. The result follows by noting that the last expression above is equal to $V_t(x_t)$. □

The next proposition shows that there is a simple solution to the optimality equation in (7). For notational brevity, in this proposition and throughout the rest of the article, we let

$$L_{it}^\alpha = \sum_{j \in \mathcal{J}} \alpha_{ijt} + \cdots + \sum_{j \in \mathcal{J}} \alpha_{ij\tau} \quad (8)$$

$$M_t^\alpha = \max_{\mathcal{S} \subset \mathcal{J}} \left\{ \sum_{j \in \mathcal{J}} P_j(\mathcal{S}) \left[r_j - \sum_{i \in \mathcal{L}} a_{ij}\, L_{i,t+1}^\alpha \right] - \sum_{i \in \mathcal{L}} \sum_{j \in \mathcal{J}} \alpha_{ijt}\, \mathbf{1}(j \in \mathcal{S})\, a_{ij} \right\}. \quad (9)$$

We note that both L_{it}^α and M_t^α are straightforward functions of the Lagrange multipliers as long as we can solve problem (9) efficiently. We are now ready to show the next proposition.

PROPOSITION 2: The solution to the optimality equation in (7) is given by

$$V_t^\alpha(x_t) = M_t^\alpha + \cdots + M_\tau^\alpha + \sum_{i \in \mathcal{L}} L_{it}^\alpha\, x_{it}.$$

PROOF: We show the result by induction over the time periods. It is easy to show the result for the last time period. Assuming that the result holds for time period $t+1$, we have $V_{t+1}^\alpha(x_t - \sum_{i \in \mathcal{L}} a_{ij}\, e_i) - V_{t+1}^\alpha(x_t) = -\sum_{i \in \mathcal{L}} L_{i,t+1}^\alpha\, a_{ij}$. Using this expression and the induction assumption in (7), we obtain

$$V_t^\alpha(x_t) = \max_{\mathcal{S} \subset \mathcal{J}} \left\{ \sum_{j \in \mathcal{J}} P_j(\mathcal{S}) \left[r_j - \sum_{i \in \mathcal{L}} L_{i,t+1}^\alpha\, a_{ij} \right] - \sum_{i \in \mathcal{L}} \sum_{j \in \mathcal{J}} \alpha_{ijt}\, \mathbf{1}(j \in \mathcal{S})\, a_{ij} \right\} + \sum_{i \in \mathcal{L}} \sum_{j \in \mathcal{J}} \alpha_{ijt}\, x_{it} + M_{t+1}^\alpha + \cdots + M_\tau^\alpha + \sum_{i \in \mathcal{L}} L_{i,t+1}^\alpha\, x_{it}.$$

The result follows by noting the definition of M_t^α in (9) and the fact that $L_{it}^\alpha = \sum_{j \in \mathcal{J}} \alpha_{ijt} + L_{i,t+1}^\alpha$. □

The optimal total expected revenue is $V_1(c)$. By Proposition 1, $V_1(c)$ is bounded from above by $V_1^\alpha(c)$ as long as the Lagrange multipliers are positive. Therefore, to obtain the tightest possible upper bound on $V_1(c)$, we can solve the problem

$$\min_{\alpha \ge 0} \left\{ V_1^\alpha(c) \right\}. \quad (10)$$

It turns out that we can obtain an optimal solution to the problem above by solving a linear program that very much resembles problem (2)–(5). To see this, we first note that

$$V_1^\alpha(c) = \sum_{t \in \mathcal{T}} M_t^\alpha + \sum_{i \in \mathcal{L}} L_{i1}^\alpha\, c_i \quad (11)$$

$$M_t^\alpha = \max_{\mathcal{S}\subset\mathcal{J}} \left\{ R(\mathcal{S}) - \sum_{i\in\mathcal{L}} Q_i(\mathcal{S})\, L_{i,t+1}^\alpha - \sum_{i\in\mathcal{L}}\sum_{j\in\mathcal{J}} \alpha_{ijt}\,\mathbf{1}(j\in\mathcal{S})\,a_{ij} \right\}, \quad (12)$$

where the first equality is by Proposition 2 and the second equality is by the definitions of M_t^α, $R(\mathcal{S})$ and $Q_i(\mathcal{S})$. In this case, the next proposition shows that the linear program

$$\zeta_{LP} = \min \sum_{t\in\mathcal{T}} \mu_t + \sum_{i\in\mathcal{L}} c_i\,\Lambda_{i1} \quad (13)$$

$$\text{subject to } \mu_t \geq R(\mathcal{S}) - \sum_{i\in\mathcal{L}} Q_i(\mathcal{S})\,\Lambda_{i,t+1} - \sum_{i\in\mathcal{L}}\sum_{j\in\mathcal{J}} \mathbf{1}(j\in\mathcal{S})\,a_{ij}\,\alpha_{ijt} \quad \forall\,\mathcal{S}\subset\mathcal{J}, t\in\mathcal{T}\setminus\{\tau\} \quad (14)$$

$$\mu_\tau \geq R(\mathcal{S}) - \sum_{i\in\mathcal{L}}\sum_{j\in\mathcal{J}} \mathbf{1}(j\in\mathcal{S})\,a_{ij}\,\alpha_{ij\tau} \quad \forall\,\mathcal{S}\subset\mathcal{J} \quad (15)$$

$$\Lambda_{it} = \sum_{j\in\mathcal{J}} \alpha_{ijt} + \ldots + \sum_{j\in\mathcal{J}} \alpha_{ij\tau} \quad \forall\, i\in\mathcal{L},\ t\in\mathcal{T} \quad (16)$$

$$\mu_t \text{ and } \Lambda_{it} \text{ are free, } \alpha_{ijt}\geq 0 \quad \forall\, i\in\mathcal{L}, j\in\mathcal{J}, t\in\mathcal{T} \quad (17)$$

is equivalent to problem (10).

PROPOSITION 3: We have $\zeta_{LP} = \min_{\alpha\geq 0}\{V_1^\alpha(c)\}$.

PROOF: If $\hat{\alpha} = \{\hat{\alpha}_{ijt} : i\in\mathcal{L}, j\in\mathcal{J}, t\in\mathcal{T}\}$ is an optimal solution to problem (10), then the definition of L_{it}^α in (8) and the definition of M_t^α in (12) imply that $\{M_t^{\hat{\alpha}} : t\in\mathcal{T}\}$, $\{L_{it}^{\hat{\alpha}} : i\in\mathcal{L}, t\in\mathcal{T}\}$, $\{\hat{\alpha}_{ijt} : i\in\mathcal{L}, j\in\mathcal{J}, t\in\mathcal{T}\}$ is a feasible solution to problem (13)–(17) with the objective value $\sum_{t\in\mathcal{T}} M_t^{\hat{\alpha}} + \sum_{i\in\mathcal{L}} c_i\, L_{i1}^{\hat{\alpha}}$. Therefore, we have $\zeta_{LP} \leq \sum_{t\in\mathcal{T}} M_t^{\hat{\alpha}} + \sum_{i\in\mathcal{L}} c_i\, L_{i1}^{\hat{\alpha}} = V_1^{\hat{\alpha}}(c) = \min_{\alpha\geq 0}\{V_1^\alpha(c)\}$, where the first equality follows from (11).

On the other hand, if $\{\hat{\mu}_t : t\in\mathcal{T}\}$, $\{\hat{\Lambda}_{it} : i\in\mathcal{L}, t\in\mathcal{T}\}$, $\{\hat{\alpha}_{ijt} : i\in\mathcal{L}, j\in\mathcal{J}, t\in\mathcal{T}\}$ is an optimal solution to problem (13)–(17), then we have $\hat{\Lambda}_{it} = L_{it}^{\hat{\alpha}}$ for all $i\in\mathcal{L}, t\in\mathcal{T}$ by constraints (16). Noting the definition of M_t^α in (12), constraints (14)–(15) together with the fact that problem (13)–(17) is a minimization problem imply that $\hat{\mu}_t = M_t^{\hat{\alpha}}$ for all $t\in\mathcal{T}$. Therefore, we have $\zeta_{LP} = \sum_{t\in\mathcal{T}} M_t^{\hat{\alpha}} + \sum_{i\in\mathcal{L}} L_{i1}^{\hat{\alpha}}\, c_i = V_1^{\hat{\alpha}}(c) \geq \min_{\alpha\geq 0}\{V_1^\alpha(c)\}$. □

We emphasize that the discussion in the proof of Proposition 3 also shows that if $\{\hat{\mu}_t : t\in\mathcal{T}\}$, $\{\hat{\Lambda}_{it} : i\in\mathcal{L}, t\in\mathcal{T}\}$, $\{\hat{\alpha}_{ijt} : i\in\mathcal{L}, j\in\mathcal{J}, t\in\mathcal{T}\}$ is an optimal solution to problem (13)–(17), then $\{\hat{\alpha}_{ijt} : i\in\mathcal{L}, j\in\mathcal{J}, t\in\mathcal{T}\}$ is an optimal solution to problem (10).

Associating the dual variables $\{y_t(\mathcal{S}) : \mathcal{S}\subset\mathcal{J}, t\in\mathcal{T}\}$ with constraints (14)–(15) and the dual variables $\{z_{it} : i\in\mathcal{L}, t\in\mathcal{T}\}$ with constraints (16), the dual of problem (13)–(17) is

$$\begin{aligned}
\zeta_{LP} = \max\ & \sum_{t\in\mathcal{T}}\sum_{\mathcal{S}\subset\mathcal{J}} R(\mathcal{S})\, y_t(\mathcal{S}) \\
\text{subject to }\ & \sum_{\mathcal{S}\subset\mathcal{J}} \mathbf{1}(j\in\mathcal{S})\,a_{ij}\,y_t(\mathcal{S}) \leq z_{i1} + \ldots + z_{it} \quad \forall\, i\in\mathcal{L}, j\in\mathcal{J}, t\in\mathcal{T} \\
& z_{i1} = c_i \quad \forall\, i\in\mathcal{L} \\
& z_{it} = -\sum_{\mathcal{S}\subset\mathcal{J}} Q_i(\mathcal{S})\, y_{t-1}(\mathcal{S}) \quad \forall\, i\in\mathcal{L}, t\in\mathcal{T}\setminus\{1\} \\
& \sum_{\mathcal{S}\subset\mathcal{J}} y_t(\mathcal{S}) = 1 \quad \forall\, t\in\mathcal{T} \\
& y_t(\mathcal{S})\geq 0,\ z_{it} \text{ is free} \quad \forall\,\mathcal{S}\subset\mathcal{J}, i\in\mathcal{L}, t\in\mathcal{T}.
\end{aligned}$$

Substituting for the decision variables $\{z_{it} : i\in\mathcal{L}, t\in\mathcal{T}\}$ by using the second and third sets of constraints, we can drop these decision variables and the problem above becomes

$$\zeta_{LP} = \max \sum_{t\in\mathcal{T}}\sum_{\mathcal{S}\subset\mathcal{J}} R(\mathcal{S})\, y_t(\mathcal{S}) \quad (18)$$

$$\text{subject to } \sum_{\mathcal{S}\subset\mathcal{J}} Q_i(\mathcal{S})\, y_1(\mathcal{S}) + \cdots + \sum_{\mathcal{S}\subset\mathcal{J}} Q_i(\mathcal{S})\, y_{t-1}(\mathcal{S}) + \sum_{\mathcal{S}\subset\mathcal{J}} \mathbf{1}(j\in\mathcal{S})\,a_{ij}\,y_t(\mathcal{S}) \leq c_i \quad \forall\, i\in\mathcal{L}, j\in\mathcal{J}, t\in\mathcal{T} \quad (19)$$

$$\sum_{\mathcal{S}\subset\mathcal{J}} y_t(\mathcal{S}) = 1 \quad \forall\, t\in\mathcal{T} \quad (20)$$

$$y_t(\mathcal{S})\geq 0 \quad \forall\,\mathcal{S}\subset\mathcal{J}, t\in\mathcal{T}. \quad (21)$$

Problem (18)–(21) is the deterministic linear program that we propose in this article. We have $\zeta_{LP} = \min_{\alpha\geq 0}\{V_1^\alpha(c)\}$ by Proposition 3 and $\min_{\alpha\geq 0}\{V_1^\alpha(c)\} \geq V_1(c)$ by Proposition 1. Therefore, similar to the optimal objective value of problem (2)–(5), the optimal objective value of problem (18)–(21) provides an upper bound on $V_1(c)$.

Problems (2)–(5) and (18)–(21) are similar to each other. As a matter of fact, the only difference between them is in the way in which they capture the capacity availabilities. Constraints (3) in problem (2)–(5) are relatively straightforward and they ensure that the total expected capacity consumptions over the planning horizon do not exceed the leg capacities.

The interpretation of constraints (19) in problem (18)–(21) is a bit more intricate. We begin by noting that the right side of the constraints

$$\mathbf{1}(j \in \mathcal{S})\, a_{ij} \le c_i - \sum_{\mathcal{S}' \subset \mathcal{J}} Q_i(\mathcal{S}')\, y_1(\mathcal{S}') - \cdots - \sum_{\mathcal{S}' \subset \mathcal{J}} Q_i(\mathcal{S}')\, y_{t-1}(\mathcal{S}') \quad \forall\, \mathcal{S} \subset \mathcal{J}, i \in \mathcal{L}, j \in \mathcal{J}, t \in \mathcal{T} \tag{22}$$

is the expected remaining capacity on flight leg i at time period t. Therefore, constraints (22) ensure that if we offer a set that includes itinerary j at time period t, then the capacity consumed by itinerary j on flight leg i should not exceed the expected remaining capacity on flight leg i. Constraints (22) can be interpreted as capacity constraints, but they apply to each time period, each itinerary, each flight leg and each set. In contrast, constraints (3) are in aggregate form in the sense that they apply only to each flight leg. If we multiply constraints (22) with $y_t(\mathcal{S})$, add over all $\mathcal{S} \subset \mathcal{J}$ and note that $\sum_{\mathcal{S} \subset \mathcal{J}} y_t(\mathcal{S}) = 1$, then we obtain constraints (19) in problem (18)–(21). This discussion suggests that constraints (19) are in a more disaggregate form than constraints (3), and hence, they may be stronger. However, in the next section, we give two examples to show that it is possible to find $\{y_t(\mathcal{S}) : \mathcal{S} \subset \mathcal{J}, t \in \mathcal{T}\}$ that satisfy constraints (19), but not constraints (3), and it is possible to find $\{h_t(\mathcal{S}) : \mathcal{S} \subset \mathcal{J}, t \in \mathcal{T}\}$ that satisfy constraints (3), but not constraints (19). Therefore, neither of constraints (3) and (19) are provably stronger. In practice, however, since constraints (19) operate at a more disaggregate level than constraints (3), the upper bounds obtained by problem (18)–(21) tend to be tighter than the upper bounds obtained by problem (2)–(5).

The number of decision variables in problem (18)–(21) increases exponentially with the number of itineraries. However, the number of constraints is $|\mathcal{L}|\,|\mathcal{J}|\,|\mathcal{T}| + |\mathcal{T}|$ and this suggests solving problem (18)–(21) by using column generation. In Section 7, we discuss solving problem (18)–(21) by using column generation under a particular choice of the probabilities $\{P_j(\mathcal{S}) : j \in \mathcal{S}, \mathcal{S} \subset \mathcal{J}\}$.

5. COMPARISON OF THE DETERMINISTIC LINEAR PROGRAMS

The optimal objective values of problems (2)–(5) and (18)–(21) both provide upper bounds on the optimal total expected revenue. In this section, we begin by presenting two examples that show that neither of these upper bounds is provably tighter than the other one. After this inconclusive result, we consider an asymptotic regime where the leg capacities and the number of time periods in the planning horizon increase linearly with the same rate. In this asymptotic regime, we establish a result that roughly shows that the upper bound obtained by problem (18)–(21) tends to be tighter than the upper bound obtained by problem (2)–(5).

Noting that Z_{LP} and ζ_{LP} are respectively the optimal objective values of problems (2)–(5) and (18)–(21), we begin with an example that shows that it is possible to have $Z_{LP} < \zeta_{LP}$. We consider a problem instance with $\mathcal{T} = \{1\}$, $\mathcal{L} = \{1\}$, $\mathcal{J} = \{1, 2\}$, $r_1 = r_2 = 10$, $c_1 = 1$ and $a_{1j} = 2$ for all $j \in \{1, 2\}$. Letting $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$ respectively be the sets $\{1\}$, $\{2\}$ and $\{1, 2\}$, we use the probabilities $P_1(\mathcal{S}_1) = 0.9$, $P_2(\mathcal{S}_2) = 0.9$, $P_1(\mathcal{S}_3) = 0.2$ and $P_2(\mathcal{S}_3) = 0.6$. Omitting the nonnegativity constraints, problem (2)–(5) for this problem instance becomes

$$\begin{aligned} Z_{LP} = \max \quad & 9\, h_1(\mathcal{S}_1) + 9\, h_1(\mathcal{S}_2) + 8\, h_1(\mathcal{S}_3) \\ \text{subject to } & 1.8\, h_1(\mathcal{S}_1) + 1.8\, h_1(\mathcal{S}_2) + 1.6\, h_1(\mathcal{S}_3) \le 1 \\ & h_1(\mathcal{S}_1) + h_1(\mathcal{S}_2) + h_1(\mathcal{S}_3) + h_1(\varnothing) = 1. \end{aligned}$$

It is easy to see that $Z_{LP} = 5$. On the other hand, problem (18)–(21) is

$$\begin{aligned} \zeta_{LP} = \max \quad & 9\, y_1(\mathcal{S}_1) + 9\, y_1(\mathcal{S}_2) + 8\, y_1(\mathcal{S}_3) \\ \text{subject to } & 2\, y_1(\mathcal{S}_1) + 2\, y_1(\mathcal{S}_3) \le 1 \\ & 2\, y_1(\mathcal{S}_2) + 2\, y_1(\mathcal{S}_3) \le 1 \\ & y_1(\mathcal{S}_1) + y_1(\mathcal{S}_2) + y_1(\mathcal{S}_3) + y_1(\varnothing) = 1. \end{aligned}$$

We have $\zeta_{LP} = 9$ so that $Z_{LP} < \zeta_{LP}$ for this problem instance.

Our second example shows that it is possible to have $Z_{LP} > \zeta_{LP}$. We consider a problem instance with $\mathcal{T} = \{1\}$, $\mathcal{L} = \{1\}$, $\mathcal{J} = \{1\}$, $r_1 = 10$, $c_1 = 1$, $a_{11} = 2$ and $P_1(\{1\}) = 0.5$. Problem (2)–(5) for this problem instance becomes

$$\begin{aligned} Z_{LP} = \max \quad & 5\, h_1(\{1\}) \\ \text{subject to } \quad & h_1(\{1\}) \le 1 \text{ and } h_1(\{1\}) + h_1(\varnothing) = 1 \end{aligned}$$

so that we have $Z_{LP} = 5$. On the other hand, problem (18)–(21) is

$$\begin{aligned} \zeta_{LP} = \max \quad & 5\, y_1(\{1\}) \\ \text{subject to } \quad & 2\, y_1(\{1\}) \le 1 \text{ and } y_1(\{1\}) + y_1(\varnothing) = 1. \end{aligned}$$

We have $\zeta_{LP} = 5/2$ so that $Z_{LP} > \zeta_{LP}$ for this problem instance.

In the remainder of this section, we consider an asymptotic regime where the leg capacities and the number of time periods in the planning horizon increase linearly with the same rate. For this purpose, we consider a family of network revenue management problems $\{\mathcal{P}^\theta : \theta \in \mathbb{Z}_+\}$ parameterized by the scaling parameter θ. Problem $\mathcal{P}^\theta$ takes place over the planning horizon $\mathcal{T}^\theta = \{1, \ldots, \theta\tau\}$ and the initial capacity on flight leg i in this problem is θc_i. All other parameters of problem $\mathcal{P}^\theta$ are the same as those described in Section 2. This is a standard way of scaling the problem in the revenue management literature to obtain asymptotic results; see [6].

We let Z_{LP}^θ and ζ_{LP}^θ respectively be the optimal objective values of problems (2)–(5) and (18)–(21) when these problems are solved with planning horizon $\mathcal{T}^\theta$ and leg capacities $\{\theta c_i : i \in \mathcal{L}\}$. The next proposition shows that $\lim_{\theta\to\infty} \zeta_{LP}^\theta / Z_{LP}^\theta \le 1$.

PROPOSITION 4: We have $\lim_{\theta\to\infty} \zeta_{LP}^\theta / Z_{LP}^\theta \le 1$.

PROOF: The dual of problem (2)–(5) is

$$\begin{aligned} Z_{LP} = \min \quad & \sum_{i\in\mathcal{L}} c_i\,\pi_i + \sum_{t\in\mathcal{T}} \sigma_t \\ \text{subject to} \quad & \sum_{i\in\mathcal{L}} Q_i(\mathcal{S})\,\pi_i + \sigma_t \ge R(\mathcal{S}) \\ & \forall\, S \subset \mathcal{J}, t \in \mathcal{T} \\ & \pi_i \ge 0,\ \sigma_t \text{ is free} \quad \forall\, i \in \mathcal{L}, t \in \mathcal{T}. \end{aligned}$$

The decision variables $\{\sigma_t : t \in \mathcal{T}\}$ take the same value $\max_{\mathcal{S}\subset\mathcal{J}}\{R(\mathcal{S}) - \sum_{i\in\mathcal{L}} Q_i(\mathcal{S})\,\pi_i\}$ in the optimal solution to the problem above. Therefore, we can replace these decision variables with a single decision variable and write the problem above as

$$Z_{LP} = \min \quad \sum_{i\in\mathcal{L}} c_i\,\pi_i + \tau\,\sigma \tag{23}$$

$$\text{subject to} \quad \sum_{i\in\mathcal{L}} Q_i(\mathcal{S})\,\pi_i + \sigma \ge R(\mathcal{S}) \quad \forall\, S \subset \mathcal{J} \tag{24}$$

$$\pi_i \ge 0,\ \sigma \text{ is free} \quad \forall\, i \in \mathcal{L}. \tag{25}$$

We let $\{\hat{\pi}_i : i \in \mathcal{L}\}$, $\hat{\sigma}$ be an optimal solution to problem (23)–(25). We note that if we solve this problem with planning horizon $\mathcal{T}^\theta$ and leg capacities $\{\theta c_i : i \in \mathcal{L}\}$, then an optimal solution to this problem is still $\{\hat{\pi}_i : i \in \mathcal{L}\}$, $\hat{\sigma}$. This implies that $Z_{LP}^\theta = \theta Z_{LP}$ and if we let $\hat{\sigma}_t = \hat{\sigma}$ for all $t \in \mathcal{T}^\theta$, then $\{\hat{\pi}_i : i \in \mathcal{L}\}$, $\{\hat{\sigma}_t : t \in \mathcal{T}^\theta\}$ is still an optimal dual solution to problem (2)–(5) when we solve this problem with planning horizon $\mathcal{T}^\theta$ and leg capacities $\{\theta c_i : i \in \mathcal{L}\}$. In this case, by using the duality theory on problem (2)–(5), we have

$$Z_{LP}^\theta = \max \quad \sum_{t\in\mathcal{T}^\theta} \sum_{\mathcal{S}\subset\mathcal{J}} R(\mathcal{S})\,h_t(\mathcal{S}) + \sum_{i\in\mathcal{L}} \hat{\pi}_i \left[\theta c_i - \sum_{t\in\mathcal{T}^\theta} \sum_{\mathcal{S}\subset\mathcal{J}} Q_i(\mathcal{S})\,h_t(\mathcal{S})\right] \tag{26}$$

$$\text{subject to (4), (5).} \tag{27}$$

We let $\hat{Q}_i = \max_{\mathcal{S}\subset\mathcal{J}} Q_i(\mathcal{S})$ for all $i \in \mathcal{L}$ and $\{\hat{y}_t(\mathcal{S}) : \mathcal{S} \subset \mathcal{J}, t \in \mathcal{T}^\theta\}$ be an optimal solution to problem (18)–(21) when we solve this problem with planning horizon $\mathcal{T}^\theta$ and leg capacities $\{\theta c_i : i \in \mathcal{L}\}$. Since $\sum_{\mathcal{S}\subset\mathcal{J}} \hat{y}_{\theta\tau}(\mathcal{S}) = 1$ and $\hat{y}_{\theta\tau}(\mathcal{S}) \ge 0$ for all $\mathcal{S} \subset \mathcal{J}$, we have

$$\begin{aligned} \sum_{t\in\mathcal{T}^\theta} \sum_{\mathcal{S}\subset\mathcal{J}} Q_i(\mathcal{S})\,\hat{y}_t(\mathcal{S}) - \hat{Q}_i &\le \sum_{t\in\mathcal{T}^\theta} \sum_{\mathcal{S}\subset\mathcal{J}} Q_i(\mathcal{S})\,\hat{y}_t(\mathcal{S}) \\ &\quad - \sum_{\mathcal{S}\subset\mathcal{J}} Q_i(\mathcal{S})\,\hat{y}_{\theta\tau}(\mathcal{S}) \le \sum_{t\in\mathcal{T}^\theta} \sum_{\mathcal{S}\subset\mathcal{J}} Q_i(\mathcal{S})\,\hat{y}_t(\mathcal{S}) \\ &\quad - \sum_{\mathcal{S}\subset\mathcal{J}} Q_i(\mathcal{S})\,\hat{y}_{\theta\tau}(\mathcal{S}) + \sum_{\mathcal{S}\subset\mathcal{J}} \mathbf{1}(j \in \mathcal{S})\,a_{ij}\,\hat{y}_{\theta\tau}(\mathcal{S}) \le \theta c_i \end{aligned} \tag{28}$$

for all $i \in \mathcal{L}$, where the third inequality follows from constraints (19) for time period $\theta\tau$ and any itinerary j. Since $\{\hat{y}_t(\mathcal{S}) : \mathcal{S} \subset \mathcal{J}, t \in \mathcal{T}^\theta\}$ is a feasible but not necessarily an optimal solution to problem (26)–(27), we obtain

$$\theta Z_{LP} = Z_{LP}^\theta \ge \sum_{t\in\mathcal{T}^\theta} \sum_{\mathcal{S}\subset\mathcal{J}} R(\mathcal{S})\,\hat{y}_t(\mathcal{S}) + \sum_{i\in\mathcal{L}} \hat{\pi}_i \left[\theta c_i - \sum_{t\in\mathcal{T}^\theta} \sum_{\mathcal{S}\subset\mathcal{J}} Q_i(\mathcal{S})\,\hat{y}_t(\mathcal{S})\right] \ge \zeta_{LP}^\theta - \sum_{i\in\mathcal{L}} \hat{\pi}_i\,\hat{Q}_i,$$

where the second inequality follows from (28) and the fact that $\hat{\pi}_i \ge 0$ for all $i \in \mathcal{L}$. The final result follows by dividing the expression above by θZ_{LP} and taking the limit. □

Therefore, we have $\lim_{\theta\to\infty}[\zeta_{LP}^\theta - Z_{LP}^\theta]/Z_{LP}^\theta \le 0$ and the percent gap between ζ_{LP}^θ and Z_{LP}^θ becomes negative as the leg capacities and the number of time periods in the planning horizon increase linearly with the same rate.

Letting $\{V_t(\cdot \mid \theta) : t \in \mathcal{T}^\theta\}$ be the value functions obtained by solving the optimality equation in (1) with planning horizon $\mathcal{T}^\theta$, Gallego et al. [4] show that $\lim_{\theta\to\infty} Z_{LP}^\theta / V_1(\theta c \mid \theta) = 1$. In other words, the percent gap between the optimal objective value of problem (2)–(5) and the optimal total expected revenue diminishes as the leg capacities and the number of time periods in the planning horizon increase linearly with the same rate. An immediate corollary to Proposition 4 is that the same property holds for the optimal objective value of problem (18)–(21).

COROLLARY 5: We have $\lim_{\theta\to\infty} \zeta_{LP}^\theta / V_1(\theta c \mid \theta) = 1$.

6. CONTROL POLICIES FROM THE DETERMINISTIC LINEAR PROGRAMS

In this section, we describe several ways in which the linear programs in Sections 3 and 4 can be used to decide which itineraries to offer at each time period.

6.1. Bid Price Policy from the Deterministic Linear Program

This is the approach described in Section 3. Letting $\{\hat{\pi}_i : i \in \mathcal{L}\}$ be the optimal values of the dual variables associated with constraints (3) in problem (2)–(5), we solve problem (6) to decide which itineraries to offer at time period t; see [9]. As mentioned before, this approach is equivalent to approximating $V_{t+1}(x_t)$ on the right side of (1) by $\tilde{V}_{t+1}(x_t) = \sum_{i \in \mathcal{L}} \hat{\pi}_i \, x_{it}$.

6.2. Decomposition from the Deterministic Linear Program

This approach decomposes the network revenue management problem into a number of single-leg revenue management problems. In particular, letting $\{\hat{\pi}_i : i \in \mathcal{L}\}$ be the optimal values of the dual variables associated with constraints (3) in problem (2)–(5), we consider the single-leg revenue management problem that takes place over flight leg i under the assumption that $r_j - \sum_{k \in \mathcal{L} \setminus \{i\}} a_{kj} \, \hat{\pi}_k$ is the revenue associated with itinerary j. We can obtain the optimal total expected revenue for this single-leg revenue management problem by solving the optimality equation

$$v_{it}(x_{it}) = \max_{\mathcal{S} \in \mathcal{O}_i(x_{it})} \left\{ \sum_{j \in \mathcal{J}} P_j(\mathcal{S}) \Big[r_j - \sum_{k \in \mathcal{L} \setminus \{i\}} a_{kj} \, \hat{\pi}_k + v_{i,t+1}(x_{it} - a_{ij}) - v_{i,t+1}(x_{it}) \Big] \right\} + v_{i,t+1}(x_{it}), \quad (29)$$

where we let $\mathcal{O}_i(x_{it}) = \{\mathcal{S} \subset \mathcal{J} : \mathbf{1}(j \in \mathcal{S}) \, a_{ij} \leq x_{it} \; \forall \, j \in \mathcal{J}\}$ and use an optimality equation that is similar to the one in (1), but focus only on flight leg i. Zhang and Adelman [11] show that

$$V_1(c) \leq v_{i1}(c_i) + \sum_{k \in \mathcal{L} \setminus \{i\}} \hat{\pi}_k \, c_k \leq Z_{LP}. \quad (30)$$

Therefore, we can solve the optimality equation in (29) to obtain an upper bound on the optimal total expected revenue that is tighter than the one provided by problem (2)–(5). In Appendix A, we give an alternative proof for the second inequality above that provides additional insight.

Repeating this approach for all $i \in \mathcal{L}$, the tightest possible upper bound on $V_1(c)$ is

$$\min_{i \in \mathcal{L}} \left\{ v_{i1}(c_i) + \sum_{k \in \mathcal{L} \setminus \{i\}} \hat{\pi}_k \, c_k \right\}.$$

Furthermore, we can collect the one-dimensional value functions $\{v_{it}(\cdot) : i \in \mathcal{L}, t \in \mathcal{T}\}$ together to construct the separable value function approximation $\tilde{V}_t(x_t) = \sum_{i \in \mathcal{L}} v_{it}(x_{it})$ for all $t \in \mathcal{T}$. In this case, we can decide which itineraries to offer at time period t by replacing $V_{t+1}(x_t)$ on the right side of (1) with $\tilde{V}_{t+1}(x_t)$ and solving this problem.

6.3. Bid Price Policy from the Alternative Deterministic Linear Program

This approach is similar to the one in Section 6.1. Letting $\hat{\alpha}$ be an optimal solution to problem (10), we replace $V_{t+1}(x_t)$ on the right side of (1) with $V^{\hat{\alpha}}_{t+1}(x_t) = M^{\hat{\alpha}}_{t+1} + \cdots + M^{\hat{\alpha}}_{\tau} + \sum_{i \in \mathcal{L}} L^{\hat{\alpha}}_{i,t+1} \, x_{it}$ and solve this problem to decide which itineraries to offer at time period t.

6.4. Decomposition from the Alternative Deterministic Linear Program

The idea behind this approach is similar to the one in Section 6.2, but this approach uses the linear program that we propose in the current paper. We let $\{\hat{\alpha}_{ijt} : i \in \mathcal{L}, j \in \mathcal{J}, t \in \mathcal{T}\}$ be the optimal values of the dual variables associated with constraints (19) in problem (18)–(21). We choose a flight leg i and relax constraints (19) for all other flight legs by associating the dual multipliers $\{\hat{\alpha}_{kjt} : k \in \mathcal{L} \setminus \{i\}, j \in \mathcal{J}, t \in \mathcal{T}\}$ with them. In this case, the objective function of problem (18)–(21) becomes

$$\sum_{t \in \mathcal{T}} \sum_{\mathcal{S} \subset \mathcal{J}} R(\mathcal{S}) \, y_t(\mathcal{S}) - \sum_{t \in \mathcal{T}} \sum_{j \in \mathcal{J}} \sum_{k \in \mathcal{L} \setminus \{i\}} \hat{\alpha}_{kjt} \times \left[\sum_{\mathcal{S} \subset \mathcal{J}} Q_k(\mathcal{S}) \, y_1(\mathcal{S}) + \cdots + \sum_{\mathcal{S} \subset \mathcal{J}} Q_k(\mathcal{S}) \, y_{t-1}(\mathcal{S}) + \sum_{\mathcal{S} \subset \mathcal{J}} \mathbf{1}(j \in \mathcal{S}) \, a_{kj} \, y_t(\mathcal{S}) - c_k \right].$$

In Appendix B, we show that simply by arranging the terms and using the definitions of $R(\mathcal{S})$, $Q_i(\mathcal{S})$ and L^{α}_{it}, the expression above can be written as

$$\sum_{t \in \mathcal{T}} \sum_{\mathcal{S} \subset \mathcal{J}} \sum_{j \in \mathcal{S}} P_j(\mathcal{S}) \left[r_j - \sum_{k \in \mathcal{L} \setminus \{i\}} a_{kj} \, L^{\hat{\alpha}}_{k,t+1} - \sum_{k \in \mathcal{L} \setminus \{i\}} [\, \hat{\alpha}_{kjt} \, \mathbf{1}(j \in \mathcal{S}) \, a_{kj} / P_j(\mathcal{S})] \right] y_t(\mathcal{S}) + \sum_{k \in \mathcal{L} \setminus \{i\}} L^{\hat{\alpha}}_{k1} \, c_k,$$

where we use the convention that $P_j(\mathcal{S})[\mathbf{1}(j \in \mathcal{S})/P_j(\mathcal{S})] = \mathbf{1}(j \in \mathcal{S})$ when $P_j(\mathcal{S}) = 0$. Therefore, the duality theory implies that the linear program

$$\zeta_{LP} = \max \quad \sum_{t \in \mathcal{T}} \sum_{\mathcal{S} \subset \mathcal{J}} \sum_{j \in \mathcal{S}} P_j(\mathcal{S}) \Big[r_j - \sum_{k \in \mathcal{L} \setminus \{i\}} a_{kj} \, L^{\hat{\alpha}}_{k,t+1} - \sum_{k \in \mathcal{L} \setminus \{i\}} [\, \hat{\alpha}_{kjt} \, \mathbf{1}(j \in \mathcal{S}) \, a_{kj} / P_j(\mathcal{S})] \Big] y_t(\mathcal{S}) + \sum_{k \in \mathcal{L} \setminus \{i\}} L^{\hat{\alpha}}_{k1} \, c_k$$

$$\text{subject to} \quad (20), (21)$$

$$\sum_{\mathcal{S} \subset \mathcal{J}} Q_i(\mathcal{S}) \, y_1(\mathcal{S}) + \cdots + \sum_{\mathcal{S} \subset \mathcal{J}} Q_i(\mathcal{S}) \, y_{t-1}(\mathcal{S}) + \sum_{\mathcal{S} \subset \mathcal{J}} \mathbf{1}(j \in \mathcal{S}) \, a_{ij} \, y_t(\mathcal{S}) \le c_i \quad \forall \, j \in \mathcal{J}, t \in \mathcal{T}$$

has the same optimal objective value as problem (18)–(21).

We consider the single-leg revenue management problem that takes place over flight leg i under the assumption that $r_j - \sum_{k \in \mathcal{L} \setminus \{i\}} a_{kj} \, L^{\hat{\alpha}}_{k,t+1} - \sum_{k \in \mathcal{L} \setminus \{i\}} [\, \hat{\alpha}_{kjt} \, \mathbf{1}(j \in \mathcal{S}) \, a_{kj} / P_j(\mathcal{S})]$ is the revenue associated with itinerary j when we offer set $\mathcal{S}$ at time period t. If we compare the last problem above with problem (18)–(21) and ignore the constant term $\sum_{k \in \mathcal{L} \setminus \{i\}} L^{\hat{\alpha}}_{k1} \, c_k$ in the objective function, then it is easy to see that the last problem above is the linear program for the single-leg revenue management problem that takes place over flight leg i. Therefore, $\zeta_{LP} - \sum_{k \in \mathcal{L} \setminus \{i\}} L^{\hat{\alpha}}_{k1} \, c_k$ is an upper bound on the optimal total expected revenue for this single-leg revenue management problem. On the other hand, we can obtain the optimal total expected revenue for the single-leg revenue management problem that takes place over flight leg i by solving the optimality equation

$$\vartheta_{it}(x_{it}) = \max_{\mathcal{S} \subset \mathcal{O}_i(x_{it})} \Big\{ \sum_{j \in \mathcal{S}} P_j(\mathcal{S}) \Big[r_j - \sum_{k \in \mathcal{L} \setminus \{i\}} a_{kj} \, L^{\hat{\alpha}}_{k,t+1} - \sum_{k \in \mathcal{L} \setminus \{i\}} [\, \hat{\alpha}_{kjt} \, \mathbf{1}(j \in \mathcal{S}) \, a_{kj} / P_j(\mathcal{S})] + \vartheta_{i,t+1}(x_{it} - a_{ij}) - \vartheta_{i,t+1}(x_{it}) \Big] \Big\} + \vartheta_{i,t+1}(x_{it}). \tag{31}$$

We have $\vartheta_{i1}(c_i) \le \zeta_{LP} - \sum_{k \in \mathcal{L} \setminus \{i\}} L^{\hat{\alpha}}_{k1} \, c_k$ by the discussion above. Furthermore, the next proposition shows that $V_1(c) \le \vartheta_{i1}(c_i) + \sum_{k \in \mathcal{L} \setminus \{i\}} L^{\hat{\alpha}}_{k1} \, c_k$. Therefore, we have

$$V_1(c) \le \vartheta_{i1}(c_i) + \sum_{k \in \mathcal{L} \setminus \{i\}} L^{\hat{\alpha}}_{k1} \, c_k \le \zeta_{LP}$$

and we can solve the optimality equation in (31) to obtain an upper bound that is tighter than the one provided by problem (18)–(21). We note that the inequality above is analogous to the one in (30).

PROPOSITION 6: Letting $\hat{\alpha} = \{\hat{\alpha}_{ijt} : i \in \mathcal{L}, j \in \mathcal{J}, t \in \mathcal{T}\}$ be the optimal values of the dual variables associated with constraints (19) in problem (18)–(21), we have $V_t(x_t) \le \vartheta_{it}(x_{it}) + \sum_{k \in \mathcal{L} \setminus \{i\}} L^{\hat{\alpha}}_{kt} \, x_{kt}$.

PROOF: We show the result by induction over the time periods. It is easy to show the result for the last time period. Assuming that the result holds for time period $t+1$, we let $\hat{\mathcal{S}}$ be an optimal solution to problem (1). We have

$$\begin{aligned}
V_t(x_t) &= \sum_{j \in \hat{\mathcal{S}}} P_j(\hat{\mathcal{S}}) \Big[r_j + V_{t+1}\Big(x_t - \sum_{i \in \mathcal{L}} a_{ij} \, e_i \Big) \Big] + \Big[1 - \sum_{j \in \mathcal{J}} P_j(\hat{\mathcal{S}}) \Big] V_{t+1}(x_t) \\
&\le \sum_{j \in \hat{\mathcal{S}}} P_j(\hat{\mathcal{S}}) \Big[r_j + \vartheta_{i,t+1}(x_{it} - a_{ij}) + \sum_{k \in \mathcal{L} \setminus \{i\}} L^{\hat{\alpha}}_{k,t+1} \, [x_{kt} - a_{kj}] \Big] + \Big[1 - \sum_{j \in \mathcal{J}} P_j(\hat{\mathcal{S}}) \Big] \times \Big[\vartheta_{i,t+1}(x_{it}) + \sum_{k \in \mathcal{L} \setminus \{i\}} L^{\hat{\alpha}}_{k,t+1} \, x_{kt} \Big] \\
&= \sum_{j \in \hat{\mathcal{S}}} P_j(\hat{\mathcal{S}}) \Big[r_j - \sum_{k \in \mathcal{L} \setminus \{i\}} a_{kj} \, L^{\hat{\alpha}}_{k,t+1} + \vartheta_{i,t+1}(x_{it} - a_{ij}) - \vartheta_{i,t+1}(x_{it}) \Big] + \vartheta_{i,t+1}(x_{it}) + \sum_{k \in \mathcal{L} \setminus \{i\}} L^{\hat{\alpha}}_{kt} \, x_{kt} - \sum_{k \in \mathcal{L} \setminus \{i\}} \sum_{j \in \mathcal{J}} \hat{\alpha}_{kjt} \, x_{kt} \\
&\le \sum_{j \in \hat{\mathcal{S}}} P_j(\hat{\mathcal{S}}) \Big[r_j - \sum_{k \in \mathcal{L} \setminus \{i\}} a_{kj} \, L^{\hat{\alpha}}_{k,t+1} + \vartheta_{i,t+1}(x_{it} - a_{ij}) - \vartheta_{i,t+1}(x_{it}) \Big] + \vartheta_{i,t+1}(x_{it}) + \sum_{k \in \mathcal{L} \setminus \{i\}} L^{\hat{\alpha}}_{kt} \, x_{kt} - \sum_{k \in \mathcal{L} \setminus \{i\}} \sum_{j \in \mathcal{J}} \hat{\alpha}_{kjt} \, \mathbf{1}(j \in \hat{\mathcal{S}}) \, a_{kj} \\
&\le \vartheta_{it}(x_{it}) + \sum_{k \in \mathcal{L} \setminus \{i\}} L^{\hat{\alpha}}_{kt} \, x_{kt},
\end{aligned}$$

where the first inequality follows from the induction assumption, the second equality follows from arranging the terms and using the definition of L^{α}_{it} in (8), the second inequality follows from the fact that $\hat{\mathcal{S}} \in \mathcal{O}(x_t)$ and $\hat{\alpha}_{ijt} \ge 0$ for all $i \in \mathcal{L}$, $j \in \mathcal{J}$ and the third inequality follows from the fact that $\hat{\mathcal{S}}$ is a feasible but not necessarily an optimal solution to problem (31). □

Similar to Section 6.2, we can repeat this approach for all $i \in \mathcal{L}$ and construct the separable value function approximation $\tilde{V}_t(x_t) = \sum_{i \in \mathcal{L}} \vartheta_{it}(x_{it})$ for all $t \in \mathcal{T}$.

7. APPLICATIONS OF THE LOGIT MODEL

The essence of the four control policies described in Section 6 is to construct approximations to the value functions and to decide which itineraries to offer by plugging the value function approximations into the right side of the optimality equation in (1). However, the number of possible values for the decision variable $\mathcal{S}$ in the optimality equation in (1) increases exponentially with the number of itineraries, and it may not be easy to decide which itineraries to offer even if we have approximations to the value functions. In this section, we begin by briefly reviewing a result shown by Gallego et al. [4] that establishes that deciding which itineraries to offer is tractable as long as the probabilities $\{P_j(\mathcal{S}) : j \in \mathcal{S}, \mathcal{S} \subset \mathcal{J}\}$ are characterized by the multinomial logit model with disjoint consideration sets. This result also implies that the column generation subproblem for problem (2)–(5) is tractable. After reviewing the result shown by Gallego et al. [4], we establish that the column generation subproblem for problem (18)–(21) can be formulated as an integer program under the multinomial logit model with disjoint consideration sets. Throughout the rest of the paper, we refer to the multinomial logit model with disjoint consideration sets simply as the logit model.

The logit model assumes that there are multiple customer types and customers of different types are interested in disjoint sets of itineraries. The set of customer types is $\mathcal{C}$. At each time period, a customer of type l arrives with probability λ_l. The set of itineraries that a customer of type l is interested in is $\mathcal{J}_l$. In other words, a customer of type l either purchases an itinerary in $\mathcal{J}_l$ or does not purchase an itinerary at all. We assume that $\mathcal{J}_l \cap \mathcal{J}_{l'} = \emptyset$ for all $l \neq l'$ so that customers of different types are interested in disjoint sets of itineraries. We use binary decision variables, rather than sets, to represent which itineraries are offered and define

$$z_j = \begin{cases} 1 & \text{if itinerary } j \text{ is offered} \\ 0 & \text{otherwise.} \end{cases}$$

We let $P_j(z)$ be the probability that a customer purchases itinerary j whenever the set of offered itineraries is given by $z = \{z_j : j \in \mathcal{J}\}$.

The logit model associates the preference weights $\{\rho_j : j \in \mathcal{J}\}$ with the itineraries. If the set of offered itineraries is given by $z = \{z_j : j \in \mathcal{J}\}$ and a customer of type l arrives, then this customer purchases itinerary j with probability $\mathbf{1}(j \in \mathcal{J}_l)\, \rho_j\, z_j / [\sum_{m \in \mathcal{J}_l} \rho_m\, z_m + \rho_0^l]$, where ρ_0^l is the strictly positive preference weight associated with purchasing nothing for customer type l. Therefore, we have

$$P_j(z) = \lambda_l \frac{\rho_j\, z_j}{\sum_{m \in \mathcal{J}_l} \rho_m\, z_m + \rho_0^l}$$

for all $j \in \mathcal{J}_l$ under the logit model.

7.1. Applications of the Logit Model to the Deterministic Linear Program

If we use the bid price policy described in Section 6.1, then we decide which itineraries to offer by solving problem (6). Under the logit model, this problem becomes

$$\max_{z \in \mathcal{Z}(x_t)} \left\{ \sum_{l \in \mathcal{C}} \sum_{j \in \mathcal{J}_l} \lambda_l \frac{\rho_j\, z_j}{\sum_{m \in \mathcal{J}_l} \rho_m\, z_m + \rho_0^l} \left[r_j - \sum_{i \in \mathcal{L}} a_{ij}\, \hat{\pi}_i \right] \right\}$$
$$= \sum_{l \in \mathcal{C}} \max_{z^l \in \mathcal{Z}^l(x_t)} \left\{ \sum_{j \in \mathcal{J}_l} \frac{\lambda_l\, \rho_j\, z_j \left[r_j - \sum_{i \in \mathcal{L}} a_{ij}\, \hat{\pi}_i \right]}{\sum_{m \in \mathcal{J}_l} \rho_m\, z_m + \rho_0^l} \right\}, \quad (32)$$

where we let $z^l = \{z_j : j \in \mathcal{J}_l\}$ and capture the set of itineraries that we can offer at time period t by $\mathcal{Z}(x_t) = \{z \in \{0,1\}^{|\mathcal{J}|} : a_{ij}\, z_j \leq x_{it} \forall\, i \in \mathcal{L}, j \in \mathcal{J}\}$ and $\mathcal{Z}^l(x_t) = \{z^l \in \{0,1\}^{|\mathcal{J}_l|} : a_{ij}\, z_j \leq x_{it} \forall\, i \in \mathcal{L}, j \in \mathcal{J}_l\}$. Gallego et al. [4] show that it is possible to obtain an optimal solution to problem (32) simply by sorting $\{r_j - \sum_{i \in \mathcal{L}} a_{ij}\, \hat{\pi}_i : j \in \mathcal{J}_l\}$ and checking the objective value obtained by $|\mathcal{J}_l| + 1$ possible solutions. Interestingly, the values of $\{\rho_j : j \in \mathcal{J}_l\}$ do not play a role in the sorting procedure. An alternative proof for this result is given in [9]. In Appendix C, we give a second alternative proof and we feel that our proof clearly shows why the values of $\{\rho_j : j \in \mathcal{J}_l\}$ do not play a role in the sorting procedure. We also note that the fact that customers of different types are interested in disjoint sets of itineraries plays a crucial role in this result. Otherwise, [3] show that problem (32) is NP-hard.

If we use the dynamic programming decomposition approach described in Section 6.2, then we replace $V_{t+1}(x_t)$ on the right side of (1) with $\sum_{i \in \mathcal{L}} v_{i,t+1}(x_{it})$ and solve this problem to decide which itineraries to offer at time period t. Under the logit model, this problem becomes

$$\max_{z \in \mathcal{Z}(x_t)} \left\{ \sum_{l \in \mathcal{C}} \sum_{j \in \mathcal{J}_l} \lambda_l \frac{\rho_j\, z_j}{\sum_{m \in \mathcal{J}_l} \rho_m\, z_m + \rho_0^l} \left[r_j + \sum_{i \in \mathcal{L}} v_{i,t+1}(x_{it} - a_{ij}) - \sum_{i \in \mathcal{L}} v_{i,t+1}(x_{it}) \right] \right\}, \quad (33)$$

which has the same structure as problem (32) and the sorting result shown by [4] continues to apply. Similarly, van Ryzin and Liu [9] show that the column generation subproblem for problem (2)–(5) has the same structure as problem (32).

7.2. Applications of the Logit Model to the Alternative Deterministic Linear Program

If we use the bid price policy described in Section 6.3, then we first need to find an optimal solution to problem (10). By the discussion in Section 4, an optimal solution to problem (10) can be obtained by solving problem (18)–(21) through column generation. Alternatively, since problem (13)–(17) is the dual of problem (18)–(21), we can solve problem (13)–(17) through constraint generation.

Constraint generation iteratively solves a master problem that has the same objective function and decision variables as problem (13)–(17), but has only a few of constraints (14)–(15). After solving the master problem, we check if any of constraints (14)–(15) is violated by the solution. If there is one such constraint, then we add this constraint to the master problem and resolve it. Specifically, letting $\{\hat{\mu}_t : t \in \mathcal{T}\}$, $\{\hat{\Lambda}_{it} : i \in \mathcal{L}, t \in \mathcal{T}\}$, $\{\hat{\alpha}_{ijt} : i \in \mathcal{L}, j \in \mathcal{J}, t \in \mathcal{T}\}$ be the solution to the current master problem, we solve the problem

$$\max_{\mathcal{S} \subset \mathcal{J}} \left\{ R(\mathcal{S}) - \sum_{i \in \mathcal{L}} Q_i(\mathcal{S})\, \hat{\Lambda}_{i,t+1} - \sum_{i \in \mathcal{L}} \sum_{j \in \mathcal{J}} \mathbf{1}(j \in \mathcal{S})\, a_{ij}\, \hat{\alpha}_{ijt} \right\} \tag{34}$$

for all $t \in \mathcal{T} \setminus \{\tau\}$ to check if any of constraints (14) is violated by this solution. Letting $\hat{\mathcal{S}}$ be an optimal solution to problem (34), if we have $R(\hat{\mathcal{S}}) - \sum_{i \in \mathcal{L}} Q_i(\hat{\mathcal{S}})\, \hat{\Lambda}_{i,t+1} - \sum_{i \in \mathcal{L}} \sum_{j \in \mathcal{J}} \mathbf{1}(j \in \hat{\mathcal{S}})\, a_{ij}\, \hat{\alpha}_{ijt} > \hat{\mu}_t$, then the constraint

$$\mu_t \geq R(\hat{\mathcal{S}}) - \sum_{i \in \mathcal{L}} Q_i(\hat{\mathcal{S}})\, \Lambda_{i,t+1} - \sum_{i \in \mathcal{L}} \sum_{j \in \mathcal{J}} \mathbf{1}(j \in \hat{\mathcal{S}})\, a_{ij}\, \alpha_{ijt}$$

is violated by the solution $\{\hat{\mu}_t : t \in \mathcal{T}\}$, $\{\hat{\Lambda}_{it} : i \in \mathcal{L}, t \in \mathcal{T}\}$, $\{\hat{\alpha}_{ijt} : i \in \mathcal{L},\ j \in \mathcal{J}, t \in \mathcal{T}\}$. We add this constraint to the master problem and resolve it. Similarly, we solve the problem

$$\max_{\mathcal{S} \subset \mathcal{J}} \left\{ R(\mathcal{S}) - \sum_{i \in \mathcal{L}} \sum_{j \in \mathcal{J}} \mathbf{1}(j \in \mathcal{S})\, a_{ij}\, \hat{\alpha}_{ij\tau} \right\}. \tag{35}$$

to check if any of constraints (15) is violated by the solution to the current master problem. Since problem (35) is a special case of problem (34) with $\hat{\Lambda}_{i,t+1} = 0$ and $\hat{\alpha}_{ijt} = \hat{\alpha}_{ij\tau}$ for all $i \in \mathcal{L}$, $j \in \mathcal{J}$, we only consider problem (34) here.

Under the logit model, problem (34) becomes

$$\max_{z \in \{0,1\}^{|\mathcal{J}|}} \left\{ \sum_{l \in \mathcal{C}} \sum_{j \in \mathcal{J}_l} \lambda_l \frac{\rho_j\, z_j}{\sum_{m \in \mathcal{J}_l} \rho_m\, z_m + \rho_0^l} \left[r_j - \sum_{i \in \mathcal{L}} a_{ij}\, \hat{\Lambda}_{i,t+1} \right] - \sum_{l \in \mathcal{C}} \sum_{j \in \mathcal{J}_l} \sum_{i \in \mathcal{L}} a_{ij}\, \hat{\alpha}_{ijt}\, z_j \right\}$$

$$= \sum_{l \in \mathcal{C}} \max_{z^l \in \{0,1\}^{|\mathcal{J}_l|}} \left\{ \sum_{j \in \mathcal{J}_l} \frac{\lambda_l\, \rho_j\, z_j \left[r_j - \sum_{i \in \mathcal{L}} a_{ij}\, \hat{\Lambda}_{i,t+1} \right]}{\sum_{m \in \mathcal{J}_l} \rho_m\, z_m + \rho_0^l} - \sum_{j \in \mathcal{J}_l} \sum_{i \in \mathcal{L}} a_{ij}\, \hat{\alpha}_{ijt}\, z_j \right\}. \tag{36}$$

We note that due to the term $\sum_{j \in \mathcal{J}_l} \sum_{i \in \mathcal{L}} a_{ij}\, \hat{\alpha}_{ijt}\, z_j$, problem (36) does not have the same structure as problem (32). Therefore, the sorting result shown by Gallego et al. [4] does not apply and it is not necessarily possible to solve this problem through a sorting procedure. However, we now show that problem (36) can be solved as a linear integer program.

The problem inside the summation on the right side of (36) is of the form

$$\max_{z \in \{0,1\}^n} \left\{ \frac{\sum_{j=1}^n \beta_j\, \rho_j\, z_j}{\sum_{m=1}^n \rho_m\, z_m + \rho_0^l} - \sum_{j=1}^n \gamma_j\, z_j \right\} \tag{37}$$

for appropriately defined values of n, $\{\beta_j : j = 1, \ldots, n\}$ and $\{\gamma_j : j = 1, \ldots, n\}$. We make the change of variables

$$w_j = \frac{z_j}{\sum_{m=1}^n \rho_m\, z_m + \rho_0^l} \quad \text{and} \quad \kappa = \frac{1}{\sum_{m=1}^n \rho_m\, z_m + \rho_0^l}$$

so that we have $\sum_{j=1}^n \rho_j\, w_j + \rho_0^l\, \kappa = 1$ by definition. In this case, the next lemma shows that problem (37) is equivalent to the nonlinear integer program

$$\max \quad \sum_{j=1}^n \beta_j\, \rho_j\, w_j - \sum_{j=1}^n \gamma_j\, z_j \tag{38}$$

$$\text{subject to} \quad \sum_{j=1}^n \rho_j\, w_j + \rho_0^l\, \kappa = 1 \tag{39}$$

$$w_j = \kappa\, z_j \qquad \forall\, j = 1, \ldots, n \tag{40}$$

$$z_j \in \{0, 1\} \qquad \forall\, j = 1, \ldots, n \tag{41}$$

$$w_j \geq 0,\ \kappa \geq 0 \qquad \forall\, j = 1, \ldots, n. \tag{42}$$

LEMMA 7: Problems (37) and (38)–(42) have the same optimal objective value and an optimal solution to one of these problems can be recovered by using an optimal solution to the other one.

PROOF: The proof follows from an argument similar to the one that is used to show Lemma 2 in Zhang and Adelman (2006). It is based on showing that given a feasible solution to one problem, we can construct a feasible solution to the other one that yields the same objective value. □

Letting B be a large number, it is easy to see that problem (38)–(42) is equivalent to the linear integer program

$$\begin{aligned} &\max \quad \sum_{j=1}^{n} \beta_j \, \rho_j \, w_j - \sum_{j=1}^{n} \gamma_j \, z_j \\ &\text{subject to} \quad (39), (41), (42) \\ &w_j \le B \, z_j \qquad \forall \, j = 1, \dots, n \\ &w_j \le \kappa \qquad \forall \, j = 1, \dots, n \\ &w_j \ge \kappa - B \, [1 - z_j] \qquad \forall \, j = 1, \dots, n. \end{aligned}$$

Noting (40), the largest value that w_j can take is κ. Since we have $\kappa \le 1/\rho_0^l$ by (39), letting $B = 1/\rho_0^l$ in the problem above suffices. Therefore, the column generation subproblem for problem (18)–(21) can be solved as a linear integer program.

If we use the bid price policy described in Section 6.3, then after solving problem (18)–(21) to obtain an optimal solution $\hat{\alpha}$ to problem (10), we compute $L_{it}^{\hat{\alpha}}$ and $M_t^{\hat{\alpha}}$ for all $i \in \mathcal{L}$, $t \in \mathcal{T}$. Noting (9), computing $M_t^{\hat{\alpha}}$ requires solving a problem that has the same structure as problem (34) and Lemma 7 continues to apply. To decide which itineraries to offer at time period t, we replace $V_{t+1}(x_t)$ on the right side of (1) with $V_{t+1}^{\hat{\alpha}}(x_t) = M_{t+1}^{\hat{\alpha}} + \cdots + M_\tau^{\hat{\alpha}} + \sum_{i \in \mathcal{L}} L_{i,t+1}^{\hat{\alpha}} \, x_{it}$ and solve this problem. Since $V_{t+1}^{\hat{\alpha}}(x_t)$ is a linear function of x_t, it is easy to see that this problem has the same structure as problem (32) and the sorting result shown by [4] continues to apply.

If we use the dynamic programming decomposition approach described in Section 6.4, then we replace $V_{t+1}(x_t)$ on the right side of (1) with $\sum_{i \in \mathcal{L}} \vartheta_{i,t+1}(x_{it})$ and solve this problem to decide which itineraries to offer at time period t. Since $\sum_{i \in \mathcal{L}} \vartheta_{i,t+1}(x_{it})$ is a separable function, this problem has the same structure as problem (33) and the sorting result shown by Gallego et al. [4] continues to apply.

8. COMPUTATIONAL EXPERIMENTS

In this section, we test the performances of the four control policies described in Section 6. We work with two sets of test problems that are all taken from [9]. The first set of test problems involve a number of parallel flight legs that operate between the same origin destination pair and the second set of test problems involve a small airline network.

Our implementations of the control policies divide the planning horizon into five equal segments and recompute the value function approximations at the beginning of each segment by using the remaining leg capacities and the remaining number of time periods in the planning horizon. We refer to the control policies described in Sections 6.1, 6.2, 6.3, and 6.4 respectively as LP, DP-LP, ALP, and DP-ALP.

8.1. Test Problems with Parallel Flight Legs

We consider three flight legs that operate between the same origin destination pair. There is an expensive and a cheap itinerary associated with each flight leg so that the number of itineraries is six. There are two customer types. The first customer type is interested only in the expensive itineraries, whereas the second customer type is interested only in the cheap itineraries. The capacities on the three flight legs are $[30, 50, 40]$ and we scale these capacities by a scalar factor to obtain test problems with different levels of congestion. We also vary the preference weights associated with purchasing nothing. All other problem parameters are the same as those in [9].

As described in Sections 3, 4, 6.2 and 6.4, we can obtain upper bounds on the optimal total expected revenue by using LP, DP-LP, ALP, and DP-ALP. Table 1 shows the upper bounds obtained by the four control policies for different test problems. In this table, the first column shows the problem characteristics by using the triplet (q, ρ_0^1, ρ_0^2), where q is the factor that we use to scale the leg capacities, and ρ_0^1 and ρ_0^2 are the preference weights associated with purchasing nothing for the two customer types. The second, third, fourth and fifth columns respectively show the upper bounds obtained by LP, DP-LP, ALP, and DP-ALP. The sixth column shows the percent gap between the upper bounds obtained by LP and ALP, whereas the seventh columns shows the percent gap between the upper bounds obtained by DP-LP, and DP-ALP. The last column shows the CPU seconds required to solve problem (18)–(21) on a Pentium IV desktop PC with 2.4 GHz CPU and 1 GB RAM running Windows XP.

Although both LP and ALP provide upper bounds on the optimal total expected revenue, the examples in Section 5 show that neither of these upper bounds is provably tighter than the other one. On the other hand, the empirical results in Table 1 indicate that the upper bounds obtained by ALP are tighter than those obtained by LP by a small but consistent margin. The percent gap between the upper bounds is more pronounced for test problems with tight leg capacities. Similarly, the upper bounds obtained by DP-ALP are tighter than the upper bounds obtained by DP-LP. We also note that the dynamic programming decomposition approach significantly tightens the upper bounds. Although we do not show these figures in Table 1, the percent gap between the upper bounds obtained by ALP and DP-ALP can be as large as 1.5%.

Table 2 shows the total expected revenues obtained by the four control policies. The second, third, fourth and fifth columns in this table respectively show the total expected revenues obtained LP, DP-LP, ALP, and DP-ALP. We obtain these total expected revenues by simulating the performances of the four control policies under 100 customer arrival trajectories. We use common random numbers when simulating the performances of different control policies. The last two

Table 1. Comparison of the upper bounds obtained by the four control policies.

Problem (q, ρ_0^1, ρ_0^2)	LP	DP-LP	ALP	DP-ALP	LP vs. ALP	DP-LP vs. DP-ALP	CPU (secs.)
$(0.6, 10^{-4}, 10^{-4})$	55,200	55,200	55,095	55,095	0.19	0.19	175
$(0.6, 1, 5)$	53,400	53,378	53,281	53,276	0.22	0.19	126
$(0.6, 5, 10)$	50,400	49,506	50,039	49,361	0.72	0.29	182
$(0.6, 10, 20)$	45,138	44,628	44,990	44,298	0.33	0.74	124
$(0.8, 10^{-4}, 10^{-4})$	67,200	67,200	67,060	67,060	0.21	0.21	156
$(0.8, 1, 5)$	65,600	65,245	65,324	65,084	0.42	0.25	108
$(0.8, 5, 10)$	59,446	59,239	59,251	58,639	0.33	1.02	86
$(0.8, 10, 20)$	47,431	46,894	47,333	46,894	0.21	0.00	33
$(1.0, 10^{-4}, 10^{-4})$	78,000	77,972	77,860	77,834	0.18	0.18	134
$(1.0, 1, 5)$	76,000	75,599	75,721	75,441	0.37	0.21	100
$(1.0, 5, 10)$	60,731	60,492	60,668	60,492	0.10	0.00	35
$(1.0, 10, 20)$	47,442	47,368	47,442	47,368	0.00	0.00	25
$(1.2, 10^{-4}, 10^{-4})$	88,800	88,467	88,611	88,341	0.21	0.14	123
$(1.2, 1, 5)$	78,117	77,731	78,117	77,731	0.00	0.00	24
$(1.2, 5, 10)$	61,038	60,905	61,038	60,905	0.00	0.00	25
$(1.2, 10, 20)$	47,442	47,438	47,442	47,438	0.00	0.00	25
$(1.4, 10^{-4}, 10^{-4})$	93,200	93,096	93,130	93,075	0.08	0.02	95
$(1.4, 1, 5)$	78,117	78,084	78,117	78,084	0.00	0.00	25
$(1.4, 5, 10)$	61,038	61,023	61,038	61,023	0.00	0.00	25
$(1.4, 10, 20)$	47,442	47,442	47,442	47,442	0.00	0.00	25

columns show the percent gap between the total expected revenues obtained by LP and ALP, and DP-LP, and DP-ALP. The results indicate that the performance of ALP is consistently superior to the performance of LP. The average performance gap between ALP and LP is about 4.5%, which is a quite significant figure in the revenue management context. The performance of DP-ALP also tends to be better than the performance of DP-LP in general, although the margin is small.

For the three flight legs in test problem $(0.6, 1, 5)$, Fig. 1 plots the bid prices used by LP and ALP as a function of the time period when the bid prices are computed at the beginning of the planning horizon. We recall that LP and ALP periodically recompute the bid prices and the bid prices naturally change when they are recomputed later in the planning horizon. The left and right charts in Fig. 1 respectively correspond to LP and ALP. The bid prices used by LP do not

Table 2. Comparison of the total expected revenues obtained by the four control policies.

Problem (q, ρ_0^1, ρ_0^2)	LP	DP-LP	ALP	DP-ALP	LP vs. ALP	DP-LP vs. DP-ALP
$(0.6, 10^{-4}, 10^{-4})$	52,529	52,587	52,733	52,770	0.39	0.35
$(0.6, 1, 5)$	48,836	52,315	51,720	52,593	5.58	0.53
$(0.6, 5, 10)$	42,366	48,756	47,794	48,879	11.36	0.25
$(0.6, 10, 20)$	37,282	43,106	42,426	43,341	12.12	0.54
$(0.8, 10^{-4}, 10^{-4})$	63,225	63,163	63,322	63,360	0.15	0.31
$(0.8, 1, 5)$	59,544	64,094	63,340	64,111	5.99	0.03
$(0.8, 5, 10)$	49,706	57,568	56,478	57,658	11.99	0.16
$(0.8, 10, 20)$	40,599	46,553	40,919	46,566	0.78	0.03
$(1.0, 10^{-4}, 10^{-4})$	73,925	75,443	75,202	75,478	1.70	0.05
$(1.0, 1, 5)$	65,428	74,137	71,704	74,095	8.75	−0.06
$(1.0, 5, 10)$	54,026	60,535	55,753	60,539	3.10	0.01
$(1.0, 10, 20)$	42,554	47,136	43,747	47,136	2.73	0.00
$(1.2, 10^{-4}, 10^{-4})$	82,191	85,563	84,300	86,130	2.50	0.66
$(1.2, 1, 5)$	72,921	77,823	74,591	77,842	2.24	0.02
$(1.2, 5, 10)$	56,010	60,982	58,103	60,982	3.60	0.00
$(1.2, 10, 20)$	43,438	47,275	45,639	47,275	4.82	0.00
$(1.4, 10^{-4}, 10^{-4})$	86,373	89,088	86,851	89,182	0.55	0.11
$(1.4, 1, 5)$	75,899	78,252	77,189	78,252	1.67	0.00
$(1.4, 5, 10)$	57,470	61,220	59,819	61,220	3.93	0.00
$(1.4, 10, 20)$	43,923	47,278	45,436	47,278	3.33	0.00

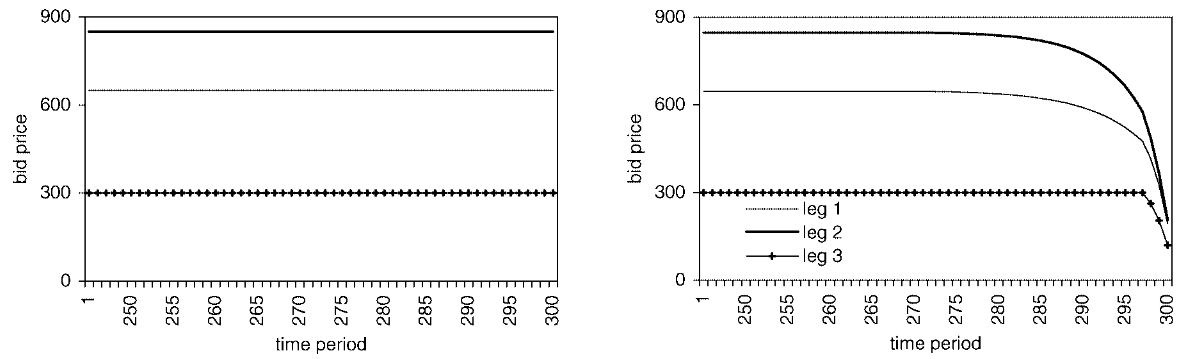

Figure 1. Bid prices used by LP and ALP as a function of the time period for test problem (0.6, 1, 5). We note that the time periods in the charts are compressed in the early portion of the planning horizon.

depend on the time period and they are very close to those used by ALP in the early portion of the planning horizon. Since the capacities are abundant in the early portion of the planning horizon, the bid prices used by ALP tend to be constant during this period. However, as expected, the bid prices used by ALP decrease as the departure time approaches and fewer opportunities to utilize the leg capacities remain.

8.2. Test Problems with an Airline Network

In this set of test problems, we consider a small airline network that connects three spokes and a hub. There are 7 flight legs, 22 itineraries and 10 customer types. Half of the itineraries are expensive and the other half are cheap. Correspondingly, half of the customer types are interested only in the expensive itineraries and the other half are interested only in the cheap itineraries. The structure of the airline network is shown in Fig. 2. All problem parameters are the same as those in [9] except for the number of time periods in the planning horizon and the leg capacities. We set $\tau = 300$ and use the leg capacities shown in Table 3. Similar to Section 8.1, we obtain different test problems by scaling the leg capacities by a scalar factor and varying the preference weights associated with purchasing nothing. We label our test problems by using the triplet (q, ρ_0^E, ρ_0^C), where q is the scaling factor for the leg capacities, and ρ_0^E and ρ_0^C are the preference weights associated with purchasing nothing for the customer types that are interested in the expensive and cheap itineraries.

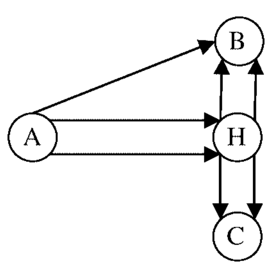

Figure 2. Structure of the airline network.

Table 4 shows the upper bounds on the optimal total expected revenues, whereas Table 5 shows the total expected revenues obtained by the four control policies. The results essentially display the same trends as those in Tables 1 and 2. For problems with tight leg capacities, the upper bounds obtained by ALP and DP-ALP are respectively tighter than the upper bounds obtained by LP and DP-LP. As the leg capacities get larger, the percent gaps between the upper bounds diminish. Comparing the total expected revenues obtained by the different control policies, the performance gap between ALP and LP can be as high as 4.1%. Furthermore, DP-ALP tends to perform better than DP-LP by a small but consistent margin in general.

9. CONCLUSIONS

We presented a new deterministic linear program for the network revenue management problem with customer choice behavior. The novel aspect of our linear program is that it naturally generates bid prices that depend on the number of time periods left until the departure time. Our linear program inherits many features of the earlier linear program used by [9]. In particular, it provides an upper bound on the optimal

Table 3. Leg capacities for the test problems with an airline network

Flight leg	Origin destination	Capacity
1	AB	30
2	AH	45
3	AH	45
4	HB	45
5	HB	45
6	HC	24
7	HC	24

Table 4. Comparison of the upper bounds obtained by the four control policies.

Problem (q, ρ_0^E, ρ_0^C)	LP	DP-LP	ALP	DP-ALP	LP vs. ALP	DP-LP vs. DP-ALP	CPU (secs.)
$(0.6, 10^{-4}, 10^{-4})$	55,800	55,738	55,597	55,537	0.37	0.36	1, 540
$(0.6, 1, 5)$	54,430	54,201	54,097	53,942	0.62	0.48	3, 065
$(0.6, 5, 10)$	49,775	49,447	49,382	49,216	0.80	0.47	2, 132
$(0.6, 10, 20)$	44,939	44,441	44,525	44,237	0.93	0.46	2, 237
$(0.8, 10^{-4}, 10^{-4})$	68,100	67,546	67,753	67,347	0.51	0.30	1, 080
$(0.8, 1, 5)$	64,819	64,447	64,523	64,301	0.46	0.23	1, 085
$(0.8, 5, 10)$	58,350	58,065	58,010	57,881	0.59	0.32	1, 167
$(0.8, 10, 20)$	49,668	49,570	49,546	49,446	0.25	0.25	891
$(1.0, 10^{-4}, 10^{-4})$	76,800	76,606	76,589	76,506	0.28	0.13	701
$(1.0, 1, 5)$	73,233	72,955	72,944	72,813	0.40	0.20	859
$(1.0, 5, 10)$	64,150	64,011	64,044	63,904	0.17	0.17	494
$(1.0, 10, 20)$	51,321	51,125	51,321	51,125	0.00	0.00	115
$(1.2, 10^{-4}, 10^{-4})$	85,200	85,036	84,989	84,935	0.25	0.12	585
$(1.2, 1, 5)$	80,229	79,778	79,991	79,686	0.30	0.12	331
$(1.2, 5, 10)$	65,321	65,212	65,321	65,212	0.00	0.00	114
$(1.2, 10, 20)$	51,321	51,308	51,321	51,308	0.00	0.00	114
$(1.4, 10^{-4}, 10^{-4})$	92,700	92,549	92,528	92,477	0.19	0.08	431
$(1.4, 1, 5)$	80,876	80,825	80,876	80,825	0.00	0.00	115
$(1.4, 5, 10)$	65,321	65,314	65,321	65,314	0.00	0.00	114
$(1.4, 10, 20)$	51,321	51,321	51,321	51,321	0.00	0.00	114

total expected revenue, it allows using the dynamic programming decomposition approach and the percent gap between its optimal objective value and the optimal total expected revenue diminishes as the leg capacities and the number of time periods in the planning horizon increase linearly with the same rate. Computational experiments indicate that our linear program can provide tighter upper bounds and the control policies that are based on our linear program can obtain higher total expected revenues.

Unfortunately, the advantages come at a cost. In particular, the number of constraints in our linear program is significantly larger than the number of constraints in the linear program that appears in the existing literature. Nevertheless, the size of our linear program is still within the capabilities of the existing computing technology. It may also be possible to aggregate some of the constraints in problem (18)–(21) to obtain linear programs that are weaker than the linear program that we propose in this paper, but still stronger than

Table 5. Comparison of the total expected revenues obtained by the four control policies.

Problem (q, ρ_0^E, ρ_0^C)	LP	DP-LP	ALP	DP-ALP	LP vs. ALP	DP-LP vs. DP-ALP
$(0.6, 10^{-4}, 10^{-4})$	50,187	52,239	52,350	52,871	4.13	1.20
$(0.6, 1, 5)$	51,100	52,924	51,522	53,029	0.82	0.20
$(0.6, 5, 10)$	46,198	48,307	46,728	48,338	1.13	0.06
$(0.6, 10, 20)$	40,552	43,379	41,886	43,283	3.18	−0.22
$(0.8, 10^{-4}, 10^{-4})$	61,853	64,884	63,053	65,659	1.90	1.18
$(0.8, 1, 5)$	60,913	63,576	61,188	63,573	0.45	0.00
$(0.8, 5, 10)$	55,098	57,003	55,670	57,074	1.03	0.12
$(0.8, 10, 20)$	46,299	48,749	46,883	48,832	1.25	0.17
$(1.0, 10^{-4}, 10^{-4})$	71,680	74,142	72,176	75,375	0.69	1.64
$(1.0, 1, 5)$	70,511	72,145	70,911	72,167	0.56	0.03
$(1.0, 5, 10)$	61,265	63,095	61,537	63,158	0.44	0.10
$(1.0, 10, 20)$	50,486	51,057	50,583	51,049	0.19	−0.02
$(1.2, 10^{-4}, 10^{-4})$	82,147	83,178	82,343	84,211	0.24	1.23
$(1.2, 1, 5)$	77,220	78,952	77,918	79,098	0.90	0.18
$(1.2, 5, 10)$	64,310	65,288	64,464	65,258	0.24	−0.05
$(1.2, 10, 20)$	51,527	51,567	51,527	51,567	0.00	0.00
$(1.4, 10^{-4}, 10^{-4})$	90,815	91,490	90,759	91,586	−0.06	0.10
$(1.4, 1, 5)$	79,895	81,116	80,183	81,130	0.36	0.02
$(1.4, 5, 10)$	65,331	65,531	65,383	65,531	0.08	0.00
$(1.4, 10, 20)$	51,650	51,650	51,650	51,650	0.00	0.00

the earlier linear program used by [9]. This is an avenue of research worth pursuing.

We emphasize that the method that we use to construct our linear program can be of interest in and of itself. The idea of relaxing certain constraints in a dynamic program by associating Lagrange multipliers with them and finding a good set of values for the Lagrange multipliers by minimizing a dual function may find applications in many different problem settings. For example, Kunnumkal and Topaloglu [5] present an application in an inventory distribution setting.

APPENDIX A

Upper Bound Obtained by the Decomposition from the Deterministic Linear Program

We let $\{\hat{\pi}_i : i \in \mathcal{L}\}$ be the optimal values of the dual variables associated with constraints (3) in problem (2)–(5). We choose a flight leg i and relax constraints (3) for all other flight legs by associating the dual multipliers $\{\hat{\pi}_k : k \in \mathcal{L} \setminus \{i\}\}$ with them. Noting the definitions of $R(\mathcal{S})$ and $Q_i(\mathcal{S})$, the duality theory implies that the linear program

$$
\begin{aligned}
Z_{LP} = \max \quad & \sum_{t \in \mathcal{T}} \sum_{\mathcal{S} \subset \mathcal{J}} \sum_{j \in \mathcal{S}} P_j(\mathcal{S}) \left[r_j - \sum_{k \in \mathcal{L} \setminus \{i\}} a_{kj}\, \hat{\pi}_k \right] h_t(\mathcal{S}) \\
& + \sum_{k \in \mathcal{L} \setminus \{i\}} \hat{\pi}_k\, c_k \\
& \text{subject to} \quad (4), (5) \\
& \sum_{t \in \mathcal{T}} \sum_{\mathcal{S} \subset \mathcal{J}} Q_i(\mathcal{S})\, h_t(\mathcal{S}) \le c_i
\end{aligned}
$$

has the same optimal objective value as problem (2)–(5).

We consider the single-leg revenue management problem that takes place over flight leg i under the assumption that $r_j - \sum_{k \in \mathcal{L} \setminus \{i\}} a_{kj}\, \hat{\pi}_k$ is the revenue associated with itinerary j. If we compare the last problem above with problem (2)–(5) and ignore the constant term $\sum_{k \in \mathcal{L} \setminus \{i\}} \hat{\pi}_k\, c_k$ in the objective function, then it is easy to see that the last problem above is the linear program for the single-leg revenue management problem that takes place over flight leg i. Therefore, $Z_{LP} - \sum_{k \in \mathcal{L} \setminus \{i\}} \hat{\pi}_k\, c_k$ is an upper bound on the optimal total expected revenue for this single-leg revenue management problem. On the other hand, we can obtain the optimal total expected revenue for the single-leg revenue management problem that takes place over flight leg i by solving the optimality equation in (29). Therefore, we have $v_{i1}(c_i) \le Z_{LP} - \sum_{k \in \mathcal{L} \setminus \{i\}} \hat{\pi}_k\, c_k$. This result is shown by Zhang and Adelman [11], but our interpretation by using a relaxation of problem (2)–(5) appears to be new and it clearly shows why we associate the revenue $r_j - \sum_{k \in \mathcal{L} \setminus \{i\}} a_{kj}\, \hat{\pi}_k$ with itinerary j.

APPENDIX B

Manipulating the Objective Function of Problem (18)–(21) after Relaxing Constraints (19)

Interchanging the order of the summations, we have

$$
\begin{aligned}
& \sum_{k \in \mathcal{L} \setminus \{i\}} \sum_{j \in \mathcal{J}} \sum_{\mathcal{S} \subset \mathcal{J}} \sum_{t \in \mathcal{T}} \hat{\alpha}_{kjt}\, Q_k(\mathcal{S}) [\, y_1(\mathcal{S}) + \cdots + y_{t-1}(\mathcal{S})] \\
& = \sum_{k \in \mathcal{L} \setminus \{i\}} \sum_{j \in \mathcal{J}} \sum_{\mathcal{S} \subset \mathcal{J}} \sum_{t \in \mathcal{T}} [\, \hat{\alpha}_{kj,t+1} + \cdots + \hat{\alpha}_{kj\tau}] Q_k(\mathcal{S})\, y_t(\mathcal{S}) \\
& = \sum_{t \in \mathcal{T}} \sum_{\mathcal{S} \subset \mathcal{J}} \sum_{k \in \mathcal{L} \setminus \{i\}} L^{\hat{\alpha}}_{k,t+1}\, Q_k(\mathcal{S})\, y_t(\mathcal{S}) \\
& = \sum_{t \in \mathcal{T}} \sum_{\mathcal{S} \subset \mathcal{J}} \sum_{j \in \mathcal{S}} \sum_{k \in \mathcal{L} \setminus \{i\}} P_j(\mathcal{S})\, a_{kj}\, L^{\hat{\alpha}}_{k,t+1}\, y_t(\mathcal{S}),
\end{aligned}
$$

where the second equality follows from the definition of L^{α}_{it} and the third equality follows from the definition of $Q_i(\mathcal{S})$. On the other hand, the definition of L^{α}_{it} implies that

$$
\sum_{k \in \mathcal{L} \setminus \{i\}} \sum_{t \in \mathcal{T}} \sum_{j \in \mathcal{J}} \hat{\alpha}_{kjt}\, c_k = \sum_{k \in \mathcal{L} \setminus \{i\}} L^{\hat{\alpha}}_{k1}\, c_k.
$$

Therefore, the expression

$$
\begin{aligned}
& \sum_{t \in \mathcal{T}} \sum_{\mathcal{S} \subset \mathcal{J}} R(\mathcal{S})\, y_t(\mathcal{S}) - \sum_{t \in \mathcal{T}} \sum_{j \in \mathcal{J}} \sum_{k \in \mathcal{L} \setminus \{i\}} \hat{\alpha}_{kjt} \left[\sum_{\mathcal{S} \subset \mathcal{J}} Q_k(\mathcal{S})\, y_1(\mathcal{S}) + \cdots \right. \\
& \qquad \left. + \sum_{\mathcal{S} \subset \mathcal{J}} Q_k(\mathcal{S})\, y_{t-1}(\mathcal{S}) + \sum_{\mathcal{S} \subset \mathcal{J}} \mathbf{1}(j \in \mathcal{S})\, a_{kj}\, y_t(\mathcal{S}) - c_k \right]
\end{aligned}
$$

can be written as

$$
\begin{aligned}
& \sum_{t \in \mathcal{T}} \sum_{\mathcal{S} \subset \mathcal{J}} \sum_{j \in \mathcal{S}} P_j(\mathcal{S})\, r_j\, y_t(\mathcal{S}) - \sum_{t \in \mathcal{T}} \sum_{\mathcal{S} \subset \mathcal{J}} \sum_{j \in \mathcal{S}} \sum_{k \in \mathcal{L} \setminus \{i\}} P_j(\mathcal{S})\, a_{kj}\, L^{\hat{\alpha}}_{k,t+1}\, y_t(\mathcal{S}) \\
& \qquad - \sum_{t \in \mathcal{T}} \sum_{\mathcal{S} \subset \mathcal{J}} \sum_{j \in \mathcal{S}} \sum_{k \in \mathcal{L} \setminus \{i\}} \hat{\alpha}_{kjt}\, \mathbf{1}(j \in \mathcal{S})\, a_{kj}\, y_t(\mathcal{S}) + \sum_{k \in \mathcal{L} \setminus \{i\}} L^{\hat{\alpha}}_{k1}\, c_k.
\end{aligned}
$$

APPENDIX C

An Alternative Proof for the Sorting Result Shown by Gallego et al. (2004)

In problem (32), we can immediately set z_j to zero when $a_{ij} > x_{it}$ for some $i \in \mathcal{L}$ and we have $z_j \in \{0, 1\}$ when $a_{ij} \le x_{it}$ for all $i \in \mathcal{L}$. Therefore, the problem inside the summation on the right side of (32) is of the form

$$
\max_{z \in \{0,1\}^n} \left\{ \frac{\sum_{j=1}^n \beta_j\, \rho_j\, z_j}{\sum_{m=1}^n \rho_m\, z_m + \rho^l_0} \right\} \tag{43}
$$

for appropriately defined values of n and $\{\beta_j : j = 1, \ldots, n\}$. The next proposition shows that problem (43) can be solved through a sorting procedure.

PROPOSITION 8: Consider problem (43) and assume without loss of generality that $\beta_1 \ge \beta_2 \ge \ldots \ge \beta_n$. There exists an optimal solution $\hat{z} = \{\hat{z}_j : j = 1, \ldots, n\}$ to this problem that satisfies

$$
\hat{z}_j = \begin{cases} 1 & \text{if } j < \hat{K} \\ 0 & \text{if } j \ge \hat{K} \end{cases} \tag{44}
$$

for an appropriately defined value of $\hat{K} \in \{1, \ldots, n+1\}$.

PROOF: As a function of ϵ, we let $g(\epsilon)$ be the optimal objective value of the linear program

$$\max \quad \frac{1}{\epsilon + \rho_0^l} \sum_{j=1}^{n} \beta_j \, \rho_j \, z_j \tag{45}$$

$$\text{subject to} \quad \sum_{j=1}^{n} \rho_j \, z_j = \epsilon \tag{46}$$

$$0 \le z_j \le 1 \quad \forall\, j = 1, \ldots, n. \tag{47}$$

It is easy to see that $g(\epsilon)$ is a continuous function of ϵ over the interval $[0, \sum_{j=1}^{n} \rho_j]$ and the optimal objective value of the problem $\max_{\epsilon \in [0, \sum_{j=1}^{n} \rho_j]} \{g(\epsilon)\}$ is equal to the optimal objective value of the continuous relaxation of problem (43). We show the final result in two steps. The first step shows that if $\epsilon = \sum_{j=1}^{\hat{K}-1} \rho_j$ for some $\hat{K} \in \{1, \ldots, n+1\}$, then there exists an optimal solution to problem (45)–(47) that has the same form as (44). The second step shows that there exists an optimal solution $\hat{\epsilon}$ to the problem $\max_{\epsilon \in [0, \sum_{j=1}^{n} \rho_j]} \{g(\epsilon)\}$ that satisfies $\hat{\epsilon} = \sum_{j=1}^{\hat{K}-1} \rho_j$ for some $\hat{K} \in \{1, \ldots, n+1\}$. These two steps show that the continuous relaxation of problem (43) has an integer optimal solution and this solution has the same form as (44).

The first step immediately follows from the fact that problem (45)–(47) is a continuous knapsack problem and an optimal solution can be found by sorting the items according to their utility to space ratios. For the second step, it is enough to show that the derivative of $g(\cdot)$ does not change sign over the interval $\left(\sum_{j=1}^{\hat{K}-1} \rho_j, \sum_{j=1}^{\hat{K}} \rho_j\right)$ for all $\hat{K} \in \{1, \ldots, n\}$. We note that if $\epsilon \in (\sum_{j=1}^{\hat{K}-1} \rho_j, \sum_{j=1}^{\hat{K}} \rho_j)$, then an optimal solution to problem (45)–(47) can be obtained by letting $\hat{z}_j = 1$ for all $j = 1, \ldots, \hat{K} - 1$, $\hat{z}_{\hat{K}} = \epsilon / \rho_{\hat{K}} - \sum_{j=1}^{\hat{K}-1} \rho_j / \rho_{\hat{K}}$ and $\hat{z}_j = 0$ for all $j = \hat{K} + 1, \ldots, n$. Therefore, we have

$$g(\epsilon) = \frac{1}{\epsilon + \rho_0^l} \left\{ \sum_{j=1}^{\hat{K}-1} \beta_j \, \rho_j + \beta_{\hat{K}} \left[\epsilon - \sum_{j=1}^{\hat{K}-1} \rho_j \right] \right\}.$$

The derivative of $g(\cdot)$ is

$$\frac{1}{\left[\epsilon + \rho_0^l\right]^2} \left\{ \beta_{\hat{K}} \, \rho_0^l - \sum_{k=1}^{\hat{K}-1} \beta_j \, \rho_j + \beta_{\hat{K}} \sum_{j=1}^{\hat{K}-1} \rho_j \right\}$$

and its sign does not depend on the value of ϵ. □

Therefore, we can find an optimal solution to problem (43) by sorting $\{\beta_j : j = 1, \ldots, n\}$ and checking $n + 1$ possible solutions. The values of $\{\rho_j : j = 1, \ldots, n\}$ do not play a role in the sorting procedure and our proof clearly shows why this is the case.

ACKNOWLEDGMENTS

The authors acknowledge the suggestions of the associate editor and three anonymous referees. The authors were supported in part by the National Science Foundation grant DMI-0422133.

REFERENCES

[1] P.P. Belobaba, Air travel demand and airline seat inventory management, Technical Report FTL-R87-8, Flight Trans Lab, Massachusetts Institute of Technology, Cambridge, MA, 1987.
[2] P.P. Belobaba and L.R. Weatherford, Comparing decis rules that incorporate customer diversion in perishable asset revenue management situations, Decis Sci 27 (1996), 343–363.
[3] J.J.M. Bront, I. Mendez-Diaz, and G. Vulcano, A column generation algorithm for choice-based network revenue management, Oper Res (2007), epub ahead of print.
[4] G. Gallego, G. Iyengar, R. Phillips, and A. Dubey, Managing flexible products on a network CORC, Technical Report TR-2004-01, Columbia University, 2004.
[5] S. Kunnumkal and H. Topaloglu, An alternative to Clark and Scarf's balance assumption for inventory distribution systems, Technical Report, Cornell University, School of Operations Research and Industrial Engineering, 2006, Available at http://legacy.orie.cornell.edu/∼huseyin/publications/publications.html.
[6] K. Talluri and G. van Ryzin, An analysis of bid-price controls for network revenue management, Management Sci 44 (1998), 1577–1593.
[7] K. Talluri and G. van Ryzin, Revenue management under a general discrete choice model of consumer behavior, Management Sci 50 (2004), 15–33.
[8] H. Topaloglu and S. Kunnumkal, Computing time-dependent bid-prices in network revenue management problems, Technical Report, Cornell University, School of Operations Research and Industrial Engineering, 2006, Available at http://legacy.orie.cornell.edu/∼huseyin/publications/publications.html.
[9] G. van Ryzin and Q. Liu, On the choice-based linear programming model for network revenue management, M&SOM 10 (2008), 288–310.
[10] G. van Ryzin and G. Vulcano, Computing virtual nesting controls for network revenue management under customer choice behavior, M&SOM (2007), epub ahead of print.
[11] D. Zhang and D. Adelman, An approximate dynamic programming approach to network revenue management with customer choice, University of Chicago, Trans Sci (2006), in review.
[12] D. Zhang and W.L. Cooper, Revenue management for parallel flights with customer choice behavior, Oper Res 53 (2005), 415–431.