Asymptotic Analysis of Exceedance Probability with Stationary Stable Steps Generated by Dissipative Flows ⋆

Uğur Tuncay Alparslan [a,*] and Gennady Samorodnitsky [b]

[a] *Department of Mathematics and Statistics, University of Nevada, Reno, Nevada 89557, USA*

[b] *School of Operations Research and Industrial Engineering, Cornell University, Ithaca, New York 14853, USA*

Abstract

We study the exceedance probability of a high threshold (ruin probability) for a random walk with a negative linear drift, where the steps of the walk (claim sizes) constitute a stationary ergodic symmetric α-stable process. We casually use the language of insurance, although this is a popular problem in many other fields of applied probability as well. We refer to ergodic theory to split the step process into two independent processes. We focus on the processes generated by dissipative flows, which are known to have a mixed moving average representation, and we restrict our attention to regular moving averages with non-negative kernels. We give results for the order of magnitude of the exceedance probability as the threshold goes to infinity in the cases of discrete-time and continuous-time claim processes.

Key words: ruin probability, stable processes, heavy tails, ergodic theory, long range dependence
1991 MSC: Primary: 60G52; secondary: 62P05

⋆ Research supported in part by NSF training grant "Graduate and Postdoctoral Training in Probability and Its Applications," and NSF Grant DMS-0303493.

* Corresponding author.
Email address: augur@unr.edu (Uğur Tuncay Alparslan).

Article published in Scandinavian Actuarial Journal 78 (2007) 1–28

1 Introduction

Given an index set $\mathbb{T}$, a stochastic process $\mathbf{S} = \{S(t),\ t \in \mathbb{T}\}$, and a non-random drift term $\underline{\mu} = \{\mu(t),\ t \in \mathbb{T}\}$, the asymptotic behavior of the *exceedance probability*,

$$\psi(u) = P\left(\sup_{t\in\mathbb{T}}(S(t) - \mu(t)) > u\right), \tag{1.1}$$

as $u \to \infty$ is a widely studied problem of applied probability theory. It has various interpretations in several different fields. In the queuing context for instance, the exceedance probability can be thought of as an overflow probability. (See for example Baccelli & Brémaud (2003).) This quantity is of fundamental interest in storage problems (see e.g. Harrison & Resnick (1976), and Brockwell, Resnick & Tweedie (1982)), and is also closely related to the tail probability of solutions to stochastic recurrence equations, including the tails of ARCH and GARCH processes (see Section 8.4 of Embrechts, Klüppelberg & Mikosch (1997)). In the context of *risk theory*, and in particular insurance, $\mathbf{S}$ can be considered as the cumulative claim size process, whereas $\underline{\mu}$ can be viewed as cumulative premium income on the insurance policy. In this case, one can view the exceedance probability as the *ruin probability with initial capital u*, or as the *ruin probability*, for short. (See Embrechts et al. (1997).)

Initially, the research on ruin probabilities focused on generalizations of the classical Cramér-Lundberg model (for details see, for instance, Embrechts et al. (1997)) with independent, identically distributed, light tailed claim sizes. More recently however, work in this area has turned to the more realistic setting of dependent claims. Furthermore, the effort by banks, insurance companies, and governmental institutions to control risk associated with extreme events resulting in "large claims" has led to the theoretical interest in modeling "heavy tailed" phenomena, presence of which is supported by empirical evidence in fields including insurance and financial markets.

The case of heavy-tailed, dependent claims also raises an important theoretical question: Can one gain more insight into the dependence structure of the heavy-tailed stationary process underlying the claims by observing the asymptotic behavior of the ruin probability? Definition of the range of dependence in its classical sense becomes ambiguous when one considers heavy-tailed processes. It is especially challenging when the second moment of the claim sizes is infinite, so that it is not possible to use covariances to quantify the strength and the range of dependence.

We have chosen the class of stationary ergodic symmetric α-stable (SαS) processes with $\alpha \in (1, 2)$ as our model for the claim sizes. Primary motivation behind this is that stable processes are, tenably, the most important class of heavy-tailed processes. Moreover, for this particular range of the parameter α, these processes have finite first moments but infinite second moments. This, concomitant with the fact that the probabilistic structure of these processes is relatively well-understood, allows us to focus on the underlying dependence structure in the presence of heavy-tails.

The setup of SαS claims with deterministic claim arrival processes and constant premium rates has been addressed in Mikosch & Samorodnitsky (2000). Based on the results of Embrechts & Veraverbeke (1982), the authors have observed that the order of magnitude of $\psi(u)$ for this model is $u^{-(\alpha-1)}$ in the case of iid claim sizes. Therefore, this is the "fastest" rate one can expect the ruin probability to decay in such a model.

The main theorem of Mikosch & Samorodnitsky (2000) allows us to analyze the asymptotic behavior of ruin probability when $\mathbb{T} = \mathbb{Z}_+$ in this setting. In this paper we apply this theorem to further extend some of the results given *ibid*, and show that for certain claim processes $\psi(u)$ decays as fast as $u^{-(\alpha-1)}$ even when the claim sizes are dependent. In the tradition of Mikosch and Samorodnitsky, we think of claim processes in this class as *short-range dependent.* We also show that for certain classes of SαS claims, $\psi(u)$ may decay slower than $u^{-(\alpha-1)}$. We think of these processes as *long-range dependent.*

In this study, we investigate the case of $\mathbb{T} = \mathbb{R}_+$ as well.

Let now our claim process, $\mathbf{X} = \{X(t),\ t \in \mathbb{T}\}$, be a measurable, stationary, ergodic SαS process with $\alpha \in (1, 2)$ given in the form

$$X(t) = \int_E f_t(x) M(dx), \quad t \in \mathbb{T}, \tag{1.2}$$

where M is a SαS random measure on a measurable space $(E, \mathcal{E})$ with a $\sigma-$finite control measure m on $\mathcal{E}$.

Since we consider *stationary* SαS processes we can choose f_t to be in a particularly descriptive form given by

$$f_t(x) = a_t(x) \left[\frac{dm \circ \phi_t}{dm}(x) \right]^{\alpha} f \circ \phi_t(x), \quad x \in E,\ t \in \mathbb{T}, \tag{1.3}$$

where $\{\phi_t\}_{t \in \mathbb{T}}$ is a *non-singular* flow, (recall that a *flow* is a family of measurable maps from E onto E such that $\phi_{t_1+t_2} = \phi_{t_1} \circ \phi_{t_2}$ for all $t_1, t_2 \in \mathbb{T}$, and ϕ_0 is the identity function on E), $\{a_t\}_{t \in \mathbb{T}}$ is a *cocycle* for this flow (i.e. for every $t_1, t_2 \in \mathbb{T}$, $a_{t_1+t_2}(x) = a_{t_2}(x) a_{t_1} \circ \phi_{t_2}(x)$ for $m-$a.a. $x \in E$) taking values in $\{-1, 1\}$, and $f \in L^\alpha(E, \mathcal{E}, m)$. (See Rosiński (1995).)

This representation is particularly important as it brings up the possibility of relating the properties of a stationary SαS process to those of a flow and a single kernel. For instance, *Hopf decomposition* (see, e.g. Krengel (1985),) of the flow $\{\phi_t\}_{t \in T}$ immediately implies that a stationary SαS process, $\mathbf{X}$, can be written (in distribution) as a sum of two independent stationary SαS processes

$$\mathbf{X} = \mathbf{X^D} + \mathbf{X^C}, \tag{1.4}$$

where $\mathbf{X^D}$ is given by representations (1.2) and (1.3) with a dissipative flow, and $\mathbf{X^C}$ is given by representations (1.2) and (1.3) with a conservative flow.

In this paper we investigate the asymptotic behavior of the ruin probability when the claims constitute a stationary SαS process generated purely by dissipative flows, i.e. processes of the form $\mathbf{X^D}$ given in (1.4).

Stationary SαS processes generated by dissipative flows are automatically ergodic and they have a *mixed moving average* representation, i.e. for any such process, $\mathbf{X} = \{X(t),\ t \in \mathbb{T}\}$, there exists a Borel space W, a σ-finite measure ν on W and a function $f \in L^\alpha(W \times \mathbb{T}, \nu \otimes \lambda)$ such that

$$\{X(t)\}_{t\in\mathbb{T}} \stackrel{d}{=} \left\{ \int_W \int_{\mathbb{T}} f(w, x-t) M(dw, dx) \right\}_{t\in\mathbb{T}}, \tag{1.5}$$

where M is a SαS random measure on the product space $W \times \mathbb{T}$ with the control measure $\nu \otimes \lambda$. Here λ is the Lebesgue measure if $\mathbb{T} = \mathbb{R}$, and the counting measure if $\mathbb{T} = \mathbb{Z}$. Moreover, if the dissipative flow itself is ergodic, (recall that a null-preserving transformation ϕ on $(E, \mathcal{E}, m)$ is called ergodic if all ϕ-invariant sets A have the property that $m(A) = 0$ or $m(A^c) = 0$), then W becomes a singleton, and the representation given by a mixed moving average reduces to the more familiar moving average,

$$\{X(t)\}_{t\in\mathbb{T}} \stackrel{d}{=} \left\{ \int_{\mathbb{T}} f(x-t) M(dx) \right\}_{t\in\mathbb{T}}. \tag{1.6}$$

See Rosiński (1995) for details.

The case of stationary SαS claims of the form $\mathbf{X^C}$ is analyzed in a separate study and the results for certain classes of these processes are presented in Alparslan & Samorodnitsky (2006).

Intuitively, one expects the range of dependence of a stationary SαS process generated by a dissipative flow to be shorter than that of a stationary SαS process generated by a conservative flow, as the flow "wanders away" in the dissipative case, unlike the conservative case. In addition, general moving average intuition tells us that the faster the kernel in the integral representation decays the less effect the history will have on the process, and hence the range of dependence will be shorter.

Indeed, the results given in Mikosch & Samorodnitsky (2000) confirm that as far as the ruin probability is concerned, stationary discrete time SαS claim sizes modelled by mixed moving averages result in the ruin probability to decay with the fastest possible rate, provided the kernel is sufficiently "nice."

In Section 2 of this paper we come up with a set of sufficient conditions for models with a discrete-time stationary dissipative SαS claim process, which guarantee that the process is short-range dependent. We also show the necessity of some of these conditions.

In Section 3 we consider the analogous problem with continuous-time stationary dissipative SαS claim sizes. In practice, this problem may arise when approximating the case where the claims are highly frequent and irregularly spaced. In addition, such a model might be of interest in the context of fluid queues and storage/dam problems.

2 Discrete time claim processes associated with an ergodic dissipative flow

Consider the ruin probability, $\psi(u)$ given by (1.1) with $\mathbb{T} = \mathbb{Z}_+$ for a model with deterministic claim arrivals, constant premium rates, and claim sizes forming a stationary SαS process generated by a dissipative flow, i.e. a process of type $\mathbf{X}^{\mathbf{D}}$ in the decomposition (1.4), with $\alpha \in (1, 2)$. This setting has been studied before in Mikosch & Samorodnitsky (2000). In their paper, the authors gave an asymptotic lower bound for the ruin probability, and showed that the order of magnitude for $\psi(u)$ is $u^{-(\alpha-1)}$, provided the kernel in the integral representation of the claim process satisfies certain conditions. In this section we further extend their work for stationary SαS claims generated by ergodic dissipative flows.

2.1 Setup and assumptions

Let the claim process $\mathbf{X} = \{X_n,\ n = 1, 2, 3, \ldots\}$ be a stationary SαS process given by

$$X_n = \int_{\mathbb{R}} f(x - n)\, M(dx), \quad n = 1, 2, 3, \ldots, \tag{2.1}$$

where $f \in L^\alpha(\mathbb{R}, \mathcal{B}, \lambda)$, and M is a SαS random measure on $(\mathbb{R}, \mathcal{B})$ with Lebesgue control measure, λ. We consider the case where $\alpha \in (1, 2)$.
Define the cumulative claim process $\mathbf{S} = \{S_n,\ n = 0, 1, 2, \ldots\}$ by

$$S_0 = 0; \quad S_n = \sum_{k=1}^{n} X_k,\ n \geq 1, \tag{2.2}$$

and let

$$h_n(x) = \sum_{k=1}^{n} f(x - k),\ n \geq 0. \tag{2.3}$$

Then it follows from Theorem 11.4.1 of Samorodnitsky & Taqqu (1994) that

$$S_n = \int_{\mathbb{R}} h_n(x) M(dx),\ n \geq 0. \tag{2.4}$$

Let $\mu > 0$ and let $\underline{\mu} = \{\mu_n = n\mu,\ n = 1, 2, 3, \ldots\}$ be a deterministic linear drift. Then the ruin probability given in (1.1) can be written as

$$\psi(u) = P\left(\sup_{n\geq 0}(S_n - \mu_n) > u\right), \quad u > 0. \tag{2.5}$$

The fundamental contribution of Mikosch & Samorodnitsky (2000) to the risk theory with stable claims, which incidentally is the origin of this work, has been to realize and prove that, under a mild condition, which will be described later in this section, the ruin probability given in (2.5) is asymptotically equivalent to another functional for large values of u, which is -from a technical point of view- "easier" to work with than $\psi(u)$. This functional is defined by

$$\psi_0(u) = \frac{C_\alpha}{2}\int_{\mathbb{R}} \sup_{n\geq 0} \frac{(h_n(x))_+^\alpha}{(u+\mu_n)^\alpha}\, dx + \frac{C_\alpha}{2}\int_{\mathbb{R}} \sup_{n\geq 0} \frac{(-h_n(x))_+^\alpha}{(u+\mu_n)^\alpha}\, dx, \quad u > 0, \tag{2.6}$$

where

$$C_\alpha = \left(\int_0^\infty x^{-\alpha}\sin x\, dx\right)^{-1}. \tag{2.7}$$

They achieved this via describing and proving the most likely way in which the ruin can occur when the claim sizes are distributed according to a stationary SαS process, by exploiting the ideas of heavy-tailed large deviations and the series representation of stable stochastic integrals. (For a detailed discussion on ψ_0 see Mikosch & Samorodnitsky (2000). For more on series representation of stable stochastic integrals see Samorodnitsky & Taqqu (1994), Section 3.10.)

In the remainder of this section we will use this result to show how certain properties of the kernel f given in (2.1) affect the asymptotic behavior of the ruin probability in the setting described above.

Before proceeding further, we define

$$\psi_0^{(\pm)}(u) = \int_{\mathbb{R}} \sup_{n\geq 0} \frac{(h_n(x))_\pm^\alpha}{(u+\mu_n)^\alpha}\, dx, \tag{2.8}$$

so that

$$\psi_0(u) = \frac{C_\alpha}{2}\left[\psi_0^{(+)}(u) + \psi_0^{(-)}(u)\right]. \tag{2.9}$$

2.2 Asymptotic analysis of the ruin probability

The impetus for the main result of this section was Braverman (2000). In the work mentioned, Braverman conjectured that, as far as the ruin probability was concerned, short-range dependence in discrete-time stationary SαS moving average processes described by (2.1) was equivalent to the kernel function in their integral representation belonging to $L^1(\mathbb{R}, \mathcal{B}, \lambda)$. Later, in Remark 2.5 however, we will give an example showing that, at least in the case where the claim sizes form a discrete-time process, this conjecture is not true. However, the following theorem confirms that there is, indeed, a connection between the range of dependence of such processes and the kernel f belonging to $L^1(\mathbb{R}, \mathcal{B}, \lambda)$.

Theorem 2.1 *Let $f : \mathbb{R} \mapsto \mathbb{R}$ given in (2.1) be a nonnegative function.*

(a) Suppose $f \notin L^1(\mathbb{R}, \mathcal{B}, \lambda)$. Then,

$$\lim_{u\to\infty} u^{\alpha-1}\psi(u) = \infty,$$

i.e. the claim process $\mathbf{X}$ is long-range dependent.

(b) Suppose for two positive integers L and R, the function defined by

$$f^*(x) := \begin{cases} \sup_{t\geq 0} f(x-t), & x \leq -L; \\ f(x), & x \in (-L, R); \\ \sup_{t\geq 0} f(x+t), & x \geq R, \end{cases} \tag{2.10}$$

is in $L^1(\mathbb{R}, \mathcal{B}, \lambda) \cap L^\alpha(\mathbb{R}, \mathcal{B}, \lambda)$. Then,

$$\lim_{u\to\infty} u^{\alpha-1}\psi(u) = \frac{C_\alpha}{2(\alpha-1)\mu} I(f) < \infty,$$

where

$$I(f) = \int_0^1 \left(\sum_{k=-\infty}^{\infty} f(x-k) \right)^\alpha dx.$$

In particular, the claim process $\mathbf{X}$ is short-range dependent.

PROOF.

(a) Start with observing that if $f \geq 0$ is not in $L^1(\mathbb{R}, \mathcal{B}, \lambda)$ then at least one of the two integrals,

$$\int_{-\infty}^1 f(x)dx, \ \int_0^\infty f(x)dx,$$

diverges. We will show that if $\int_{-\infty}^1 f(x)dx = \infty$, the ruin probability, $\psi(u)$ decays slower than $u^{-(\alpha-1)}$, by first showing that $\psi_0^{(+)}(u)$ decays slower than $u^{-(\alpha-1)}$. The conclusion is the same when $\int_{-\infty}^1 f(x)dx = \infty$, and the proof for this case is analogous to the previous case.

So now assume $\int_{-\infty}^{1} f(x)\,dx = \infty$. Note that

$$\begin{aligned}\psi_0^{(+)}(u) &= \int_{\mathbb{R}} \sup_{n\geq 0} \left(\frac{\sum_{k=1}^{n} f(x-k)}{u+n\mu} \right)^{\alpha} dx \\ &= \sum_{j=-\infty}^{\infty} \int_0^1 \sup_{n\geq 0} \left(\frac{\sum_{k=1}^{n} f(x+j-k)}{u+n\mu} \right)^{\alpha} dx \\ &= \sum_{j=-\infty}^{\infty} \int_0^1 \sup_{n\geq 0} \left(\frac{\sum_{k=j+1}^{n+j} f(x-k)}{u+n\mu} \right)^{\alpha} dx.\end{aligned} \tag{2.11}$$

But,

$$\begin{aligned}\sum_{j=-\infty}^{\infty} \int_0^1 \sup_{n\geq 0} \left(\frac{\sum_{k=j+1}^{n+j} f(x-k)}{u+n\mu} \right)^{\alpha} dx &\geq \sum_{j=-\infty}^{-1} \int_0^1 \sup_{n\geq 0} \left(\frac{\sum_{k=j+1}^{n+j} f(x-k)}{u+n\mu} \right)^{\alpha} dx \\ &\geq \sum_{j=-\infty}^{-1} \int_0^1 \sup_{n\geq 0} \left(\frac{\sum_{k=0}^{n+j} f(x-k)}{u+n\mu} \right)^{\alpha} dx,\end{aligned}$$

and substituting $n = -2j$ for $j \leq -1$ we have

$$\psi_0^{(+)}(u) \geq \sum_{j=-\infty}^{-1} \int_0^1 \left(\frac{\sum_{k=0}^{-j} f(x-k)}{u-2j\mu} \right)^{\alpha} dx = \sum_{j=1}^{\infty} \int_0^1 \left(\frac{\sum_{k=0}^{j} f(x-k)}{u+2j\mu} \right)^{\alpha} dx. \tag{2.12}$$

For any $x \in [0,1]$, define $K(x) := \sum_{k=0}^{\infty} f(x-k)$. Let

$$A := \{x \in [0,1] :\ K(x) < \infty\},$$

and $A^c := [0,1] \setminus A$. Recall $f \geq 0$. Thus, given $x \in A$, for any $\varepsilon \in (0,1)$, there exists $j_\varepsilon(x) < \infty$ such that $\sum_{k=0}^{j_\varepsilon(x)} f(x-k) \geq (1-\varepsilon)K(x)$.

Moreover if $x \in A^c$, then for all $M \in (0,\infty)$, there exists $j_M < \infty$ such that $\sum_{k=0}^{j_M} f(x-k) \geq M$.

Now, it follows from (2.12) that

$$
\begin{aligned}
\liminf_{u\to\infty} u^{\alpha-1}\psi_0^{(+)}(u) &\geq \liminf_{u\to\infty} u^{\alpha-1}\int_0^1 \sum_{j=1}^{\infty}\left(\frac{\sum_{k=0}^{j} f(x-k)}{u+2j\mu}\right)^{\alpha} dx \\
&\geq \int_0^1 \liminf_{u\to\infty} u^{\alpha-1}\sum_{j=1}^{\infty}\left(\frac{\sum_{k=0}^{j} f(x-k)}{u+2j\mu}\right)^{\alpha} dx \qquad \text{(by Fatou's lemma)} \\
&\geq \int_A \liminf_{u\to\infty} u^{\alpha-1}\sum_{j=j_\varepsilon(x)}^{\infty}\left(\frac{(1-\varepsilon)K(x)}{u+2j\mu}\right)^{\alpha} dx \\
&\quad + \int_{A^c} \liminf_{u\to\infty} u^{\alpha-1}\sum_{j=j_M}^{\infty}\left(\frac{M}{u+2j\mu}\right)^{\alpha} dx \\
&\geq \frac{1}{2\mu(\alpha-1)}\int_A [(1-\varepsilon)K(x)]^{\alpha} dx \\
&\quad + \frac{1}{2\mu(\alpha-1)}\int_{A^c} M^{\alpha} dx.
\end{aligned}
\tag{2.13}
$$

Consequently, if $\lambda(A^c)=0$ then using Hölder's inequality we have

$$
\begin{aligned}
\liminf_{u\to\infty} u^{\alpha-1}\psi_0^{(+)}(u) &\geq \frac{(1-\varepsilon)^{\alpha}}{2\mu(\alpha-1)}\left(\int_A K(x)dx\right)^{\alpha} \\
&= \frac{(1-\varepsilon)^{\alpha}}{2\mu(\alpha-1)}\left(\int_0^1 K(x)dx\right)^{\alpha} \\
&= \frac{(1-\varepsilon)^{\alpha}}{2\mu(\alpha-1)}\left(\int_{-\infty}^1 f(x)dx\right)^{\alpha} = \infty.
\end{aligned}
\tag{2.14}
$$

If on the other hand, $\lambda(A^c)>0$ then for every $M>0$,

$$
\begin{aligned}
\liminf_{u\to\infty} u^{\alpha-1}\psi_0^{(+)}(u) &\geq \frac{1}{2\mu(\alpha-1)}\int_{A^c} M^{\alpha} dx \\
&= \frac{M^{\alpha}\lambda(A^c)}{2\mu(\alpha-1)}.
\end{aligned}
\tag{2.15}
$$

Thus we conclude that

$$
\lim_{u\to\infty} u^{\alpha-1}\psi_0(u) = \frac{C_\alpha}{2}\lim_{u\to\infty} u^{\alpha-1}\psi_0^{(+)}(u) = \infty. \tag{2.16}
$$

But then it immediately follows from Remark 2.10 of Mikosch & Samorodnitsky (2000) that

$$
\lim_{u\to\infty} u^{\alpha-1}\psi(u) = \infty. \tag{2.17}
$$

(b) In the first part of the proof we will show that as u tends to infinity, the ruin probability $\psi(u)$ is indeed asymptotically equivalent to $\psi_0(u)$ given by (2.6). Once we establish this, we will work with $\psi_0(u)$ to get the desired result.

Before proceeding, observe that f^* is monotone increasing on $(-\infty, -L]$ and is monotone decreasing on $[R, \infty)$, and is the smallest of any function $g \geq f$ with this property.

We start with the following lemma:

Lemma 2.2 *If the assumptions of Theorem 2.1(b) hold then*

$$I(f^*) = \int_0^1 \left(\sum_{k=-\infty}^{\infty} f^*(x-k) \right)^\alpha dx < \infty.$$

PROOF. Since $f^* \in L^1(\mathbb{R}, \mathcal{B}, \lambda)$ is non-negative, there exists a finite integer $K > \max\{L, R\} + 1$ such that for any $x \in [0,1]$,

$$\int_{-\infty}^{x-(K-1)} f^*(y)dy + \int_{x+K-1}^{\infty} f^*(y)dy < 1. \tag{2.18}$$

It follows from the monotonicity of f^* that

$$\begin{aligned} 1 &> \int_{-\infty}^{x-(K-1)} f^*(y)dy + \int_{x+K-1}^{\infty} f^*(y)dy \\ &= \sum_{k=-\infty}^{-K} \int_0^1 f^*(x+k+y)dy + \sum_{k=K-1}^{\infty} \int_0^1 f^*(x+k+y)dy \\ &\geq \sum_{k=-\infty}^{-K} \int_0^1 f^*(x+k+0)dy + \sum_{k=K-1}^{\infty} \int_0^1 f^*(x+k+1)dy \\ &= \sum_{k=-\infty}^{-K} f^*(x+k) + \sum_{k=K}^{\infty} f^*(x+k), \end{aligned} \tag{2.19}$$

Consequently, by (2.19), Hölder's inequality, and the fact that $f^* \in L^\alpha(\mathbb{R}, \mathcal{B}, \lambda)$,

$$\begin{aligned} I(f^*) &\leq \int_0^1 \left(1 + \sum_{k=-K+1}^{K-1} f^*(x+k) \right)^\alpha dx \\ &\leq 2^{\alpha-1} \int_0^1 \left[1 + \left(\sum_{k=-K+1}^{K-1} f^*(x+k) \right)^\alpha \right] dx \\ &\leq 2^{\alpha-1} \left[1 + (2K-1)^{\alpha-1} \int_0^1 \sum_{k=-K+1}^{K-1} [f^*(x+k)]^\alpha dx \right] \\ &\leq 2^{\alpha-1} \left[1 + (2K-1)^{\alpha-1} \|f^*\|^\alpha_{L^\alpha(\mathbb{R},\mathcal{B},\lambda)} \right] < \infty. \end{aligned} \tag{2.20}$$

$\square$

The following lemma will let us work with $\psi_0(u)$ to analyze the asymptotic behavior of the ruin probability:

Lemma 2.3 *Under the assumptions of Theorem 2.1(b), relation $\psi(u) \sim \psi_0(u)$ holds as $u \to \infty$.*

PROOF. Let K be the integer as in (2.18). Via suitable changes of variables we see that for any $n \geq 1$,

$$h_n(y) \leq \min \left\{ \sum_{k=-\infty}^{-K} f^*(y+K-1+k), \ \sum_{k=K}^{\infty} f^*(y-K-n+k) \right\}. \tag{2.21}$$

Note by monotonicity of f^*, for $k \leq -K$,

$$g_1(y) := f^*(y+K-1+k)$$

is monotone increasing on $y \in (-\infty, 2-K]$. Then for any $y \in (-\infty, 2-K]$, it follows from (2.19) that

$$h_n(y) \leq \sum_{k=-\infty}^{-K} f^*(y+K-1+k) \leq \sum_{k=-\infty}^{-K} f^*(1+k) < 1. \tag{2.22}$$

Also by monotonicity of f^*, for $k \geq K$,

$$g_2(y) := f^*(y-K-n+k)$$

is monotone decreasing on $y \in [K+n, \infty)$. Then for any $y \in [K+n, \infty)$, it follows from (2.19) that

$$h_n(y) \leq \sum_{k=K}^{\infty} f^*(y-K-n+k) \leq \sum_{k=K}^{\infty} f^*(k) < 1. \tag{2.23}$$

Thus by (2.22) and (2.23) we see that for $y \in \mathbb{R} \setminus (2-K, K+n)$, $h_n(y) \leq 1$. Defining

$$m_n = \|h_n\|_{L^\alpha(\mathbb{R},\mathcal{B},\lambda)}, \tag{2.24}$$

(note that m_n is the scaling parameter of S_n), we have

$$\begin{aligned}
m_n^\alpha &= \int_{-\infty}^{2-K} h_n^\alpha(x)dx + \int_{2-K}^{K+n} h_n^\alpha(x)dx + \int_{K+n}^{\infty} h_n^\alpha(x)dx \\
&\leq \int_{-\infty}^{2-K} h_n(x)\,dx + \int_{2-K}^{K+n} h_n^\alpha(x)dx + \int_{K+n}^{\infty} h_n(x)\,dx \\
&\leq 2n\|f\|_{L^1(\mathbb{R},\mathcal{B},\lambda)} + \int_{2-K}^{K+n} h_n^\alpha(x)dx \\
&= 2n\|f\|_{L^1(\mathbb{R},\mathcal{B},\lambda)} + \sum_{j=2-K}^{K+n-1} \int_0^1 \left(\sum_{k=1-j}^{n-j} f(x-k) \right)^\alpha dx \\
&\leq 2n\|f^*\|_{L^1(\mathbb{R},\mathcal{B},\lambda)} + (2K+n-2)I(f^*).
\end{aligned} \tag{2.25}$$

Since $I(f^*) < \infty$ by Lemma 2.2 we see that $m_n = O(n^{1/\alpha})$ as $n \to \infty$. The desired result follows from Theorem 2.5 of Mikosch & Samorodnitsky (2000). $\square$

In the rest of the proof our methodology will be to split $\psi_0^{(+)}(u)$ into several pieces and treat each one separately.

Now write

$$
\begin{aligned}
\psi_0^{(+)}(u) = \int_{-\infty}^{-L} \sup_{n\geq 0} \left(\frac{\sum_{k=1}^{n} f(x-k)}{u+n\mu} \right)^{\alpha} dx + \int_{-L}^{R+1} \sup_{n\geq 0} \left(\frac{\sum_{k=1}^{n} f(x-k)}{u+n\mu} \right)^{\alpha} dx \\
+ \int_{R+1}^{\infty} \sup_{n\geq 0} \left(\frac{\sum_{k=1}^{n} f(x-k)}{u+n\mu} \right)^{\alpha} dx \\
:= \psi_1^{(+)}(u) + \psi_2^{(+)}(u) + \psi_3^{(+)}(u).
\end{aligned}
\tag{2.26}
$$

We immediately observe that

$$
\psi_2^{(+)}(u) \leq u^{-\alpha} \sum_{j=-L}^{R} \int_0^1 \left(\sum_{k=1-j}^{\infty} f(x-k) \right)^{\alpha} dx \leq u^{-\alpha}(R+L+1)\, I(f^*),
\tag{2.27}
$$

and hence, it follows from Lemma 2.2 that

$$
\psi_2^{(+)}(u) = o\left(u^{-(\alpha-1)}\right), \quad \text{as } u \to \infty.
\tag{2.28}
$$

Further write,

$$
\begin{aligned}
\psi_1^{(+)}(u) \leq \int_{-\infty}^{-L} \sup_{n\geq -x} \left(\frac{\sum_{k=1}^{n} f(x-k)}{u+n\mu} \right)^{\alpha} dx \\
+ \int_{-\infty}^{-L} \sup_{0\leq n\leq -x} \left(\frac{\sum_{k=1}^{n} f(x-k)}{u+n\mu} \right)^{\alpha} dx \\
:= \psi_{11}^{(+)}(u) + \psi_{12}^{(+)}(u).
\end{aligned}
\tag{2.29}
$$

Note that by monotonicity of f^* on $(-\infty, -L]$ we have

$$\begin{aligned}\psi_{11}^{(+)}(u) &\le \int_{-\infty}^{-L} \sup_{n \ge -x} \left(\frac{\sum_{k=1}^{n} f^*(x-k)}{u + n\mu} \right)^{\alpha} dx \le \int_{-\infty}^{-L} \left(\frac{\sum_{k=1}^{\infty} f^*(x-k)}{u - x\mu} \right)^{\alpha} dx \\ &\le \int_{-\infty}^{-L} \left(\frac{\sum_{k=-\infty}^{\lceil x \rceil - 1} f^*(k)}{u - x\mu} \right)^{\alpha} dx \le \int_{-\infty}^{-L} \left(\frac{\int_{-\infty}^{\lceil x \rceil} f^*(y)dy}{u - x\mu} \right)^{\alpha} dx.\end{aligned} \tag{2.30}$$

Moreover, since $f^* \in L^1(\mathbb{R}, \mathcal{B}, \lambda)$, for every $\varepsilon > 0$ there exists $x_\varepsilon \in (L, \infty)$ such that $\int_{-\infty}^{\lceil -x_\varepsilon \rceil} f^*(y)dy < \varepsilon^{1/\alpha}$. Then it follows from (2.30) that

$$\begin{aligned}\psi_{11}^{(+)}(u) &\le \int_{-\infty}^{-x_\varepsilon} \left(\frac{\int_{-\infty}^{\lceil x \rceil} f^*(y)dy}{u - x\mu} \right)^{\alpha} dx + \int_{-x_\varepsilon}^{-L} \left(\frac{\int_{-\infty}^{\lceil x \rceil} f^*(y)dy}{u - x\mu} \right)^{\alpha} dx \\ &\le \varepsilon \int_{-\infty}^{-x_\varepsilon} (u - x\mu)^{-\alpha} dx + u^{-\alpha}(x_\varepsilon - L) \|f^*\|_{L^1(\mathbb{R},\mathcal{B},\lambda)}^{\alpha}.\end{aligned} \tag{2.31}$$

Consequently, we see that

$$\limsup_{u \to \infty} u^{\alpha-1} \psi_{11}^{(+)}(u) \le \frac{\varepsilon}{(\alpha - 1)\mu}, \tag{2.32}$$

and letting $\varepsilon \downarrow 0$, we conclude that

$$\psi_{11}^{(+)}(u) = o\left(u^{-(\alpha-1)}\right), \quad \text{as } u \to \infty. \tag{2.33}$$

Additionally, notice that by monotonicity of f^*

$$\psi_{12}^{(+)}(u) \le \int_{-\infty}^{-L} \sup_{0 \le n \le -x} \left(\frac{n f^*(x)}{u + n\mu} \right)^{\alpha} dx \le \int_{-\infty}^{-L} \left(\frac{-x f^*(x)}{u - x\mu} \right)^{\alpha} dx. \tag{2.34}$$

Furthermore, since $f^* \in L^1(\mathbb{R}, \mathcal{B}, \lambda)$ and is monotone decreasing on $(-\infty, -L]$,

$$f^*(x) = o(-1/x), \quad \text{as } x \to -\infty,$$

and thus for every $\epsilon > 0$ there exists $x_\epsilon \in (-\infty, -L)$ such that for every

$x < x_\epsilon$, $-xf^*(x) < \epsilon^{1/\alpha}$. Therefore, it follows from (2.34) that

$$\begin{aligned}\psi_{12}^{(+)}(u) &\le \int_{-\infty}^{-x_\epsilon} \left(\frac{-xf^*(x)}{u - x\mu}\right)^\alpha dx + \int_{-x_\epsilon}^{-L} \left(\frac{-xf^*(x)}{u - x\mu}\right)^\alpha dx \\ &\le \epsilon \int_{-\infty}^{-x_\epsilon} (u - x\mu)^{-\alpha}\, dx + u^{-\alpha} \int_{-x_\epsilon}^{-L} (-xf^*(x))^\alpha\, dx \\ &\le \frac{(u + x_\epsilon \mu)^{1-\alpha}\epsilon}{(\alpha-1)\mu} + u^{-\alpha}(x_\epsilon - L)x_\epsilon \|f^*\|^\alpha_{L^\alpha(\mathbb{R},\mathcal{B},\lambda)}.\end{aligned} \tag{2.35}$$

Hence,

$$\limsup_{u\to\infty} u^{\alpha-1}\psi_{12}^{(+)}(u) \le \frac{\epsilon}{(\alpha-1)\mu}. \tag{2.36}$$

Letting $\epsilon \downarrow 0$, we see that

$$\psi_{12}^{(+)}(u) = o\left(u^{-(\alpha-1)}\right), \quad \text{as } u \to \infty, \tag{2.37}$$

and combining (2.29), (2.33), (2.37) we conclude

$$\psi_{1}^{(+)}(u) = o\left(u^{-(\alpha-1)}\right), \quad \text{as } u \to \infty \tag{2.38}$$

Next pick an arbitrary integer $M > R + 1$, and write

$$\begin{aligned}\psi_3^{(+)}(u) \le \int_M^\infty \sup_{n<x-M} \left(\frac{\sum_{k=1}^n f(x-k)}{u+n\mu}\right)^\alpha dx + \int_M^\infty \sup_{n\ge x-M} \left(\frac{\sum_{k=1}^n f(x-k)}{u+n\mu}\right)^\alpha dx \\ + \int_{R+1}^M \sup_{n\ge 0} \left(\frac{\sum_{k=1}^n f(x-k)}{u+n\mu}\right)^\alpha dx \\ := \psi_{31}^{(+)}(u) + \psi_{32}^{(+)}(u) + \psi_{33}^{(+)}(u)\end{aligned} \tag{2.39}$$

We will need the following lemma to proceed:

Lemma 2.4 *For $x > M$,*

$$\sup_{0\le n<x-M} \frac{\sum_{k=1}^n f^*(x-k)}{u+n\mu} \le \frac{\sum_{k=1}^{\lceil x-M\rceil} f^*(x-k)}{u+(x-M)\mu}. \tag{2.40}$$

PROOF. For $x > M$ and $0 \le n_0 < x - M$, it follows from monotonicity of

f^* on $[R,\infty)$ that,

$$\begin{aligned}
\sum_{k=1}^{n_0} f^*(x-k) - \frac{u+n_0\mu}{u+\lceil x-M\rceil\mu} \sum_{k=1}^{\lceil x-M\rceil} f^*(x-k) \\
&= \left[1 - \frac{u+n_0\mu}{u+\lceil x-M\rceil\mu}\right] \sum_{k=1}^{n_0} f^*(x-k) \\
&\qquad - \frac{u+n_0\mu}{u+\lceil x-M\rceil\mu} \sum_{k=n_0+1}^{\lceil x-M\rceil} f^*(x-k) \\
&\le \left[\frac{(\lceil x-M\rceil - n_0)\mu}{u+\lceil x-M\rceil\mu}\right] n_0 f^*(x-n_0) \\
&\qquad - \frac{u+n_0\mu}{u+\lceil x-M\rceil\mu}(\lceil x-M\rceil - n_0) f^*(x-n_0-1) \\
&\le \frac{-u(\lceil x-M\rceil - n_0)}{u+\lceil x-M\rceil\mu} f^*(x-n_0-1) \le 0,
\end{aligned} \tag{2.41}$$

and hence

$$\frac{1}{u+n_0\mu} \sum_{k=1}^{n_0} f^*(x-k) \le \frac{1}{u+\lceil x-M\rceil\mu} \sum_{k=1}^{\lceil x-M\rceil} f^*(x-k)\,, \tag{2.42}$$

giving the desired result. $\square$

Now, it immediately follows from Lemma 2.4 that

$$\psi_{31}^{(+)}(u) \le \int_M^\infty \left(\frac{\sum_{k=1}^{\lceil x-M\rceil} f^*(x-k)}{u+(x-M)\mu}\right)^\alpha dx \le \sum_{j=M}^\infty \int_0^1 \left(\frac{\sum_{k=1-j}^{\lceil x-M\rceil} f^*(x-k)}{u+(x+j-M)\mu}\right)^\alpha dx. \tag{2.43}$$

Defining $I_M(f^*) := \int_0^1 \left(\sum_{k=-\infty}^{1-M} f^*(x-k)\right)^\alpha dx$, we see from (2.43) that

$$\psi_{31}^{(+)}(u) \le I_M(f^*) \sum_{j=0}^\infty (u+j\mu)^{-\alpha}. \tag{2.44}$$

Furthermore, via appropriate change of variables we notice that

$$\psi_{32}^{(+)}(u) \le \int_M^\infty \left(\frac{\sum_{k=1}^\infty f(x-k)}{u+(x-M)\mu}\right)^\alpha dx \le I(f) \sum_{j=0}^\infty (u+j\mu)^{-\alpha}. \tag{2.45}$$

Lastly, it is easy to see that

$$\psi_{33}^{(+)}(u) \leq u^{-\alpha}(M - R - 1)I(f). \tag{2.46}$$

Combining (2.39), (2.44)- (2.46), we have

$$\limsup_{u\to\infty} u^{\alpha-1}\psi_3^{(+)}(u) \leq \frac{1}{(\alpha-1)\mu}[I_M(f^*) + I(f)]. \tag{2.47}$$

Letting $M \to \infty$,

$$\limsup_{u\to\infty} u^{\alpha-1}\psi_3^{(+)}(u) \leq \frac{I(f)}{(\alpha-1)\mu} < \infty. \tag{2.48}$$

Thus (2.26), (2.38), (2.28), and (2.48) give

$$\limsup_{u\to\infty} u^{\alpha-1}\psi_0^{(+)}(u) \leq \frac{I(f)}{(\alpha-1)\mu}\,, \tag{2.49}$$

and hence by (2.9), Lemma 2.3, and non-negativity of f we conclude that

$$\limsup_{u\to\infty} u^{\alpha-1}\psi(u) \leq \frac{C_\alpha}{2(\alpha-1)\mu}\, I(f) < \infty. \tag{2.50}$$

The desired result follows by observing that

$$\liminf_{u\to\infty} u^{\alpha-1}\psi(u) \geq \frac{C_\alpha}{2(\alpha-1)\mu}\, I(f), \tag{2.51}$$

by Theorem 4.1.(a) of Mikosch & Samorodnitsky (2000). $\square$

Remark 2.5 For part (b) of Theorem 2.1, merely assuming $f \in L^1(\mathbb{R}, \mathcal{B}, \lambda)$ as Braverman (2000) conjectured, is not sufficient to conclude that $\psi(u) = O\left(u^{-(\alpha-1)}\right)$ as $u \to \infty$. Consider the following example:

Example 2.6 Let $p > 2$ and let

$$f(x) = \lfloor x \rfloor \cdot \mathbf{1}\{x \geq 1,\ x - \lfloor x \rfloor \in [0, \lfloor x \rfloor^{-p})\},$$

where $\mathbf{1}\{\cdot\}$ is the indicator function, i.e. for any proposition A,

$$\mathbf{1}\{A\} = \begin{cases} 1, \text{ if } A \text{ is true;} \\ 0, \text{ otherwise.} \end{cases}$$

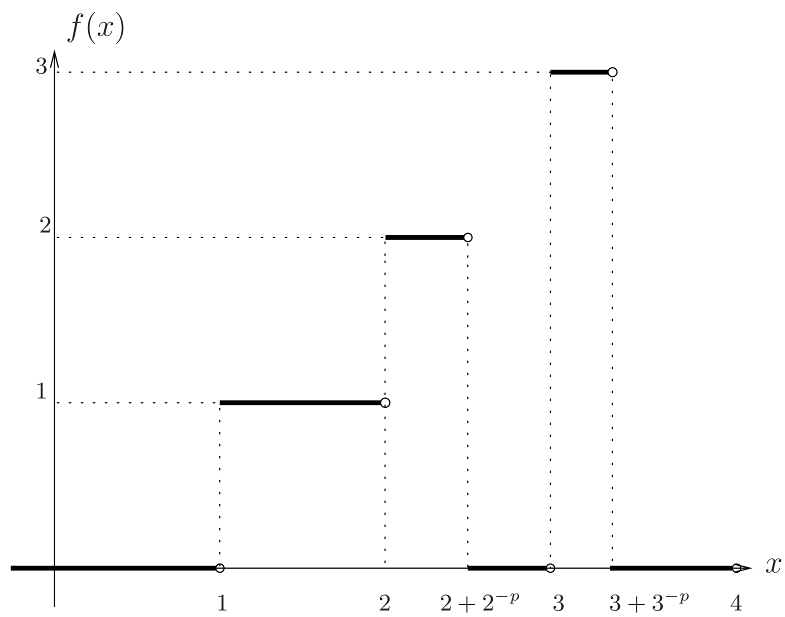

Fig. 1. Example 2.6

Note that

$$\|f\|_{L^1(\mathbb{R},\mathcal{B},\lambda)} = \sum_{k=1}^{\infty} k^{-(p-1)} < \infty. \tag{2.52}$$

Next observe that

$$\psi_0^{(+)}(u) = \sum_{i=2}^{\infty} \int_0^1 \sup_{0 \leq n \leq i-1} \left(\frac{\sum\limits_{k=i-n}^{i-1} k \cdot 1_{\{x < k^{-p}\}}}{u + n\mu} \right)^{\alpha} dx, \tag{2.53}$$

and hence

$$\begin{aligned}
\psi_0^{(+)}(u) &\geq \sum_{i=1}^{\infty} \int_0^1 \left(\frac{\sum_{k=1}^{i} k \cdot 1_{\{x<k^{-p}\}}}{u+i\mu} \right)^{\alpha} dx \\
&= \sum_{i=1}^{\infty} (u+i\mu)^{-\alpha} \sum_{j=1}^{\infty} \int_{(j+1)^{-p}}^{j^{-p}} \left(\sum_{k=1}^{i} k \cdot 1_{\{x<k^{-p}\}} \right)^{\alpha} dx \\
&\geq \sum_{i=1}^{\infty} (u+i\mu)^{-\alpha} \sum_{j=i+1}^{\infty} \int_{(j+1)^{-p}}^{j^{-p}} \left(\sum_{k=1}^{i} k \cdot 1_{\{x<k^{-p}\}} \right)^{\alpha} dx \\
&= \sum_{i=1}^{\infty} (u+i\mu)^{-\alpha} \sum_{j=i+1}^{\infty} \int_{(j+1)^{-p}}^{j^{-p}} \left(\sum_{k=1}^{i} k \right)^{\alpha} dx \\
&= \sum_{i=1}^{\infty} (u+i\mu)^{-\alpha} \left(\frac{i(i+1)}{2} \right)^{\alpha} \sum_{j=i+1}^{\infty} \int_{(j+1)^{-p}}^{j^{-p}} dx \\
&= \sum_{i=1}^{\infty} (u+i\mu)^{-\alpha} \left(\frac{i(i+1)}{2} \right)^{\alpha} (i+1)^{-p} \\
&\geq 2^{-(\alpha+p)} \sum_{i=1}^{\infty} (u+i\mu)^{-\alpha} i^{2\alpha-p}.
\end{aligned} \tag{2.54}$$

Without loss of generality take $\mu = 1$, and define

$$g(x) := (u+x)^{-\alpha} x^{2\alpha-p}. \tag{2.55}$$

Pick $p \in (\alpha+1, 2\alpha)$ and $u \geq 1$ sufficiently large.

Now, g is monotone increasing on $\left(1, \frac{(2\alpha-p)u}{p-\alpha}\right)$ and is monotone decreasing on $\left(\frac{(2\alpha-p)u}{p-\alpha}, \infty\right)$. Consequently, by the continuity and nonnegativity of g on $(0, \infty)$, one can show that

$$g(1) + \int_1^{\infty} g(x)dx \leq \sum_{i=1}^{\infty} g(i) + g\left(\frac{(2\alpha-p)u}{p-\alpha} \right). \tag{2.56}$$

Consequently, (2.54) implies

$$\begin{aligned}
u^{\alpha-1}\psi_0^{(+)}(u) &\geq 2^{-(\alpha+p)} u^{\alpha-1} \left[g(1) - g\left(\frac{(2\alpha-p)u}{p-\alpha} \right) + \int_1^{\infty} g(x)dx \right] \\
&\geq 2^{-(2\alpha+p)} u^{-1} - \frac{(p-\alpha)^{p-\alpha}(2\alpha-p)^{2\alpha-p}}{2^{(\alpha+p)}\alpha^{\alpha}} u^{2\alpha-p-1} \\
&\qquad + 2^{-(\alpha+p)} u^{\alpha-1} \int_u^{\infty} (u+x)^{-\alpha} x^{2\alpha-p} dx \\
&\sim 2^{-(\alpha+p)} u^{2\alpha-p} \int_1^{\infty} (1+y)^{-\alpha} y^{2\alpha-p} dy \quad \text{as } u \to \infty.
\end{aligned} \tag{2.57}$$

Since we picked $p \in (\alpha+1, 2\alpha)$, we see that the integral on the right hand side converges, and hence

$$\liminf_{u\to\infty} u^{\alpha-1}\psi_0^{(+)}(u) \geq \liminf_{u\to\infty} Cu^{2\alpha-p}\int_1^\infty (1+y)^{-\alpha}y^{2\alpha-p}dy = \infty. \tag{2.58}$$

3 Continuous time stationary SαS processes associated with ergodic dissipative flows

In this section we consider a continuous-time claim process $\mathbf{X}$. The continuous-time model in the insurance context is of interest as an approximation in the presence of high-frequency claims which are irregularly spaced. The model can also be applied in the context of fluid queues and storage/dam processes.

3.1 Setup and assumptions

Let the claim process $\mathbf{X} = \{X(t),\ t \geq 0\}$ be a continuous-time measurable stationary $S\alpha S$ process given by

$$X(t) = \int_{\mathbb{R}} f(x-t)\, M(dx), \quad t \geq 0, \tag{3.1}$$

where $f \in L^\alpha(\mathbb{R}, \mathcal{B}, \lambda)$, and M is a SαS random measure on $(\mathbb{R}, \mathcal{B})$ with Lebesgue control measure, λ, and $\alpha \in (1,2)$. Also for a positive constant $\mu > 0$ let the cumulative premium process be a non-random linear drift $\underline{\mu} = \{\mu(t) = t\mu,\ t \geq 0\}$. Next let

$$S(t) := \int_0^t X(s)\, ds, \quad t \geq 0, \tag{3.2}$$

be the total amount claimed until time t and define

$$h_t(x) := \int_0^t f(x-s)\, ds. \tag{3.3}$$

It follows from Theorem 11.3.2 of Samorodnitsky & Taqqu (1994) that the process

$$\mathbf{S} = \{S(t),\ t \in [0,\infty)\}$$

described by (3.2) is well-defined, and by Theorem 11.4.1 of the same text

$$S(t) = \int_{\mathbb{R}} h_t(x)\, M(dx), \quad t \geq 0. \tag{3.4}$$

Now, for $u > 0$ write the ruin probability as

$$\psi(u) = P\left(\sup_{t\geq 0}(S(t) - \mu(t)) > u\right), \quad u > 0. \tag{3.5}$$

In Braverman (2004), the author investigates the asymptotic behavior of the probabilistic tails of subadditive functionals acting on sample paths of discrete and continuous time symmetric stable processes with negative drift. In the context of the risk theory, his key result is a generalization of the main result given in Mikosch & Samorodnitsky (2000), which established the asymptotic equivalence between (2.5) and (2.6). Applied to the setting of this section, the main theorem of Braverman (2004) says that for large values of u, under certain conditions, which will be checked later in this section, the ruin probability given by (3.5) is asymptotically equivalent to the functional given by

$$\psi_0(u) := \frac{C_\alpha}{2}\int_{\mathbb{R}} \sup_{t\geq 0} \frac{(h_t(x))_+^\alpha}{(u+t\mu)^\alpha}\,dx + \frac{C_\alpha}{2}\int_{\mathbb{R}} \sup_{t\geq 0} \frac{(-h_t(x))_+^\alpha}{(u+t\mu)^\alpha}\,dx, \quad u>0, \tag{3.6}$$

where C_α is as in (2.7). In showing this, Braverman also uses the series representation of stable stochastic integrals. (See Samorodnitsky & Taqqu (1994), Section 3.10.)

In the rest of this section we will utilize this result in a similar fashion to the previous section to derive conclusions about the interplay between properties of the kernel f in the moving average representation (3.1) and the asymptotic behavior of the ruin probability (3.5).

3.2 Asymptotic behavior of the ruin probability

The following proposition is a corollary of Theorem 4.1 of Braverman (2004), and will later be used in the proof of the main result of this section.

Proposition 3.1 *If for some $\beta \in (0,1)$, the scaling parameter of $S(t)$ is $O(t^\beta)$ as t tends to infinity, i.e. if*

$$\|h_t(\cdot)\|_{L^\alpha(\mathbb{R},\mathcal{B},\lambda)} = O(t^\beta) \quad \text{as } t\to\infty, \tag{3.7}$$

then $\psi(u) \sim \psi_0(u)$ as $u\to\infty$.

PROOF. We start by proving the following lemma:

Lemma 3.2 *If for some $\beta\in(0,1)$,*

$$\|h_t(\cdot)\|_{L^\alpha(\mathbb{R},\mathcal{B},\lambda)} = O(t^\beta) \quad \text{as } t\to\infty,$$

then there exists $\tilde\varepsilon \in (0,1)$ such that the process $\tilde{\mathbf{Y}} = (\tilde Y(t),\ t\geq 0)$ defined by

$$\tilde Y(t) := (t+1)^{\tilde\varepsilon-1} S(t), \quad t\geq 0,$$

is a.s. bounded

PROOF. Since

$$\|h_t(\cdot)\|_{L^\alpha(\mathbb{R},\mathcal{B},\lambda)} = O(t^\beta) \quad \text{as } t \to \infty \text{ for some } \beta \in (0,1), \tag{3.8}$$

it follows from Proposition 7.4 of Braverman (2004) that there exists $\varepsilon_0 > 0$ such that the process

$$\left\{(n+1)^{\varepsilon_0 - 1} S(n),\ n = 0, 1, 2, \ldots\right\}$$

is a.s. bounded. Further, note by the stationarity of $\mathbf{X}$, for any $\tilde{\varepsilon} \in (0,1)$,

$$\begin{aligned}
&P\left(\sup_{n=0,1,2,\ldots} \frac{\sup_{n \le t \le n+1} |S(t) - S(n)|}{(n+1)^{1-\tilde{\varepsilon}}} \ge \lambda\right) \\
&\qquad \le \sum_{n=0}^{\infty} P\left(\sup_{n \le t \le n+1} |S(t) - S(n)| \ge \lambda (n+1)^{1-\tilde{\varepsilon}}\right) \\
&\qquad = \sum_{n=0}^{\infty} P\left(\sup_{0 \le t \le 1} |S(t)| \ge \lambda (n+1)^{1-\tilde{\varepsilon}}\right) \\
&\qquad \le \sum_{n=0}^{\infty} P\left(\int_0^1 |X(s)|\, ds \ge \lambda (n+1)^{1-\tilde{\varepsilon}}\right)
\end{aligned} \tag{3.9}$$

But it follows from Theorem 11.5.1 of Samorodnitsky & Taqqu (1994) that

$$\sum_{n=0}^{\infty} P\left(\int_0^1 |X(s)|\, ds \ge \lambda (n+1)^{1-\tilde{\varepsilon}}\right) \le C \sum_{n=0}^{\infty} \left[\lambda (n+1)^{1-\tilde{\varepsilon}}\right]^{-\alpha}, \tag{3.10}$$

for some positive constant C. Hence, we see that for any $\tilde{\varepsilon} < (1 - \alpha^{-1})$,

$$\lim_{\lambda \to \infty} P\left(\sup_{n=0,1,2,\ldots} \frac{\sup_{n \le t \le n+1} |S(t) - S(n)|}{(n+1)^{1-\tilde{\varepsilon}}} \ge \lambda\right) = 0. \tag{3.11}$$

Consequently, for any such $\tilde{\varepsilon}$, it follows from monotone convergence theorem that the process

$$\left\{(n+1)^{\tilde{\varepsilon}-1} \sup_{n \le t \le n+1} |S(t) - S(n)|,\ n = 0, 1, 2, \ldots\right\}$$

is a.s. bounded. Desired result follows by picking $\tilde{\varepsilon} \in (0, \min\{\varepsilon_0, (1 - \alpha^{-1})\})$ and observing that

$$\begin{aligned}
\sup_{t \ge 0} |\tilde{Y}(t)| \le &\sup_{n=0,1,2,\ldots} (n+1)^{\tilde{\varepsilon}-1} |S(n)| \\
&+ \sup_{n=0,1,2,\ldots} (n+1)^{\tilde{\varepsilon}-1} \sup_{n \le t \le n+1} |S(t) - S(n)|.
\end{aligned} \tag{3.12}$$

□

Now, to finish the proof of the proposition, pick $\tilde{\varepsilon} > 0$ such that $\tilde{\mathbf{Y}}$ is almost surely bounded. (The above lemma guarantees the existence of $\tilde{\varepsilon}$.) Next, define a process $\mathbf{Y} = \{Y(t),\ t \geq 0\}$ by

$$Y(t) = \frac{[\log(t\mu + 2)]^{1+\varepsilon}}{t\mu + 2} S(t), \quad t \geq 0.$$

Note that for any $\varepsilon > 0$,

$$\frac{[\log(t\mu + 2)]^{1+\varepsilon}}{t\mu + 2} = o\left((t+1)^{\tilde{\varepsilon}-1}\right) \quad \text{as } t \to \infty.$$

Then, since $\tilde{\varepsilon} > 0$ is picked such that $\tilde{\mathbf{Y}}$ is a.s. bounded, we see that for any $\varepsilon > 0$, $\mathbf{Y}$ is a.s. bounded. The claim of the proposition follows from Theorem 4.1 and Remark 4.2 of Braverman (2004). $\square$

The theorem we state below is our main result for this section. It shows that, as far as the ruin probability is concerned, there is a strong relation between long-range dependence in continuous-time stationary SαS processes generated by ergodic dissipative flows and integrability of the kernel function in their integral representation.

Theorem 3.3 *Let $f : \mathbb{R} \mapsto \mathbb{R}$ given in (3.1) be a nonnegative function.*

(a) Let $f \notin L^1(\mathbb{R}, \mathcal{B}, \lambda)$. Then, $\lim_{u\to\infty} u^{\alpha-1}\psi(u) = \infty$.
(b) Suppose for some positive reals L and R, $f^*(\cdot)$, defined as in (2.10), is in $L^1(\mathbb{R}, \mathcal{B}, \lambda) \cap L^\alpha(\mathbb{R}, \mathcal{B}, \lambda)$. Then,

$$\lim_{u\to\infty} u^{\alpha-1}\psi(u) = \frac{C_\alpha \|f\|^\alpha_{L^1(\mathbb{R},\mathcal{B},\lambda)}}{2(\alpha-1)\mu}.$$

PROOF.
(a) Start by noting that for the ruin probability defined in (3.5) and $u > 0$,

$$\psi(u) \geq P\left[\sup_{n\geq 0} (S(n) - n\mu) > u\right] := \psi_d(u). \tag{3.13}$$

Also note

$$\begin{aligned} S(n) &= \int_0^n X(n)ds = \int_{\mathbb{R}} \left(\int_0^n f(x-s)ds\right) M(dx) \\ &= \int_{\mathbb{R}} \left(\sum_{k=1}^n \int_{-1}^0 f(x-s-k)ds\right) M(dx). \end{aligned} \tag{3.14}$$

Defining

$$g(x) := \int_{-1}^0 f(x-s)ds,$$

it is easy to see that $f \notin L^1(\mathbb{R}, \mathcal{B}, \lambda)$ implies $g \notin L^1(\mathbb{R}, \mathcal{B}, \lambda)$.

Moreover, with these definitions, we observe that $\psi_d(u)$ is nothing but the ruin probability for a discrete time stationary $S\alpha S$ claim process given by

$$X_d(n) := \int_{\mathbb{R}} g(x-n)\, M(dx), \quad n = 1, 2, 3, \ldots. \tag{3.15}$$

Thus the desired result follows from (3.13), Theorem 2.1(a), and the fact that $g \notin L^1(\mathbb{R}, \mathcal{B}, \lambda)$.

(b) Define

$$\psi_0^{(+)}(u) = \int_{\mathbb{R}} \sup_{t\geq 0} \frac{(h_t(x))_+^\alpha}{(u+t\mu)^\alpha}\, dx, \quad u > 0. \tag{3.16}$$

Then, since f is nonnegative, $\psi_0(u) = \frac{C_\alpha}{2}\psi_0^{(+)}(u)$.

We will first show that

$$\lim_{u\to\infty} u^{\alpha-1}\psi_0^{(+)}(u) = \frac{\|f\|^\alpha_{L^1(\mathbb{R},\mathcal{B},\lambda)}}{(\alpha-1)\mu}. \tag{3.17}$$

Then it will suffice to show that $\psi(u) \sim \psi_0(u)$ as $u \to \infty$.

To show (3.17), we proceed in two steps. First we show that

$$\liminf_{u\to\infty} u^{\alpha-1}\psi_0^{(+)}(u) \geq \frac{\|f\|^\alpha_{L^1(\mathbb{R},\mathcal{B},\lambda)}}{(\alpha-1)\mu}. \tag{3.18}$$

Fix $z \in \mathbb{R}$. Since $\|f\|_{L^1(\mathbb{R},\mathcal{B},\lambda)} \leq \|f^*\|_{L^1(\mathbb{R},\mathcal{B},\lambda)} < \infty$, for all $\varepsilon \in (0,1)$ there exists $x_{z,\varepsilon} \in (z, \infty)$ such that

$$\int_z^{x_{z,\varepsilon}} f(y)\, dy \geq (1-\varepsilon)^{1/\alpha} \int_z^\infty f(y)\, dy.$$

Thus,

$$\begin{aligned}
\liminf_{u\to\infty} u^{\alpha-1}\psi_0^{(+)}(u) &= \liminf_{u\to\infty} u^{\alpha-1}\int_{\mathbb{R}} \sup_{t\geq 0}\left(\frac{\int_{x-t}^x f(y)\,dy}{u+t\mu}\right)^\alpha dx \\
&\geq \liminf_{u\to\infty} u^{\alpha-1}\int_{x_{z,\varepsilon}}^\infty \left(\frac{\int_z^x f(y)\,dy}{u+(x-z)\mu}\right)^\alpha dx \\
&\geq \left(\int_z^{x_{z,\varepsilon}} f(y)\,dy\right)^\alpha \liminf_{u\to\infty} u^{\alpha-1}\int_{x_{z,\varepsilon}}^\infty (u+(x-z)\mu)^{-\alpha}\,dx \\
&\geq \frac{(1-\varepsilon)}{(\alpha-1)\mu}\left(\int_z^\infty f(y)\,dy\right)^\alpha,
\end{aligned} \tag{3.19}$$

and letting $\varepsilon \downarrow 0$ and $z \downarrow -\infty$ gives (3.18).

In the second step we show

$$\limsup_{u\to\infty} u^{\alpha-1}\psi_0^{(+)}(u) \leq \frac{\|f\|^\alpha_{L^1(\mathbb{R},\mathcal{B},\lambda)}}{(\alpha-1)\mu}, \tag{3.20}$$

and to do this we follow a similar approach to that of the proof of Theorem 2.1(b). So start with writing

$$\begin{aligned}\psi_0^{(+)}(u) &= \int_{-\infty}^{-L} \sup_{t\geq 0}\left(\frac{\int_0^t f(x-s)\,ds}{u+t\mu}\right)^{\alpha} dx + \int_{-L}^{R} \sup_{t\geq 0}\left(\frac{\int_0^t f(x-s)\,ds}{u+t\mu}\right)^{\alpha} dx \\ &\quad + \int_{R}^{\infty} \sup_{t\geq 0}\left(\frac{\int_0^t f(x-s)\,ds}{u+t\mu}\right)^{\alpha} dx \\ &:= \psi_1^{(+)}(u) + \psi_2^{(+)}(u) + \psi_3^{(+)}(u),\end{aligned} \tag{3.21}$$

and immediately note that

$$\psi_2^{(+)}(u) \leq u^{-\alpha}(R+L)\|f\|^{\alpha}_{L^1(\mathbb{R},\mathcal{B},\lambda)},$$

implying

$$\psi_2^{(+)}(u) = o\left(u^{-(\alpha-1)}\right) \quad \text{as } u\to\infty. \tag{3.22}$$

Next write

$$\begin{aligned}\psi_1^{(+)}(u) &= \int_{-\infty}^{-L} \sup_{t\geq -x}\left(\frac{\int_0^t f(x-s)\,ds}{u+t\mu}\right)^{\alpha} dx \\ &\quad + \int_{-\infty}^{-L} \sup_{0\leq t\leq -x}\left(\frac{\int_0^t f(x-s)\,ds}{u+t\mu}\right)^{\alpha} dx \\ &:= \psi_{11}^{(+)}(u) + \psi_{12}^{(+)}(u).\end{aligned} \tag{3.23}$$

Since $f\in L^1(\mathbb{R},\mathcal{B},\lambda)$, for all $\theta>0$ there exists $x_\theta > L$ such that

$$\int_{-\infty}^{-x_\theta} f(y)dy < \theta^{1/\alpha}.$$

Then,

$$\begin{aligned}\limsup_{u\to\infty} u^{\alpha-1}\psi_{11}^{(+)}(u) &\leq \limsup_{u\to\infty} u^{\alpha-1}\int_{-\infty}^{-x_\theta}\left(\frac{\int_{-\infty}^{x} f(y)\,dy}{u-x\mu}\right)^{\alpha} dx \\ &\quad + \limsup_{u\to\infty} u^{\alpha-1}\int_{-x_\theta}^{-L}\left(\frac{\int_{-\infty}^{x} f(y)\,dy}{u}\right)^{\alpha} dx \\ &\leq \theta \limsup_{u\to\infty} u^{\alpha-1}\int_{-\infty}^{-x_\theta}(u-x\mu)^{-\alpha}\,dx \\ &\quad + \limsup_{u\to\infty} u^{-1}(x_\theta - L)\|f\|^{\alpha}_{L^1(\mathbb{R},\mathcal{B},\lambda)} \\ &= \frac{\theta}{(\alpha-1)\mu},\end{aligned} \tag{3.24}$$

and letting $\theta\downarrow 0$, we see that

$$\psi_{11}^{(+)}(u) = o\left(u^{-(\alpha-1)}\right) \quad \text{as } u\to\infty. \tag{3.25}$$

Moreover, by the same argument given for (2.35), for all $\epsilon > 0$ there exists $x_\epsilon \in (-\infty, -L)$ such that for every $x < x_\epsilon$, $-xf^*(x) < \epsilon^{1/\alpha}$. Thus, by monotonicity of f^*,

$$\begin{aligned}
\limsup_{u\to\infty} u^{\alpha-1}\psi_{12}^{(+)}(u) &\le \limsup_{u\to\infty} u^{\alpha-1}\int_{-\infty}^{-L} \sup_{0\le t\le -x}\left(\frac{tf^*(x)}{u+t\mu}\right)^{\alpha} dx \\
&= \limsup_{u\to\infty} u^{\alpha-1}\int_{-\infty}^{-x_\epsilon}\left(\frac{-xf^*(x)}{u-x\mu}\right)^{\alpha} dx \\
&\qquad + \limsup_{u\to\infty} u^{\alpha-1}\int_{-x_\epsilon}^{-L}\left(\frac{-xf^*(x)}{u-x\mu}\right)^{\alpha} dx \\
&\le \epsilon \limsup_{u\to\infty} u^{\alpha-1}\int_{-\infty}^{-x_\epsilon}(u-x\mu)^{-\alpha}\, dx \\
&\qquad + \limsup_{u\to\infty} u^{-1}\int_{-x_\epsilon}^{-L}(-xf^*(x))^{\alpha}\, dx \\
&\le \epsilon \limsup_{u\to\infty} u^{\alpha-1}\frac{(u+x_\epsilon\mu)^{1-\alpha}}{(\alpha-1)\mu} \\
&\qquad + \limsup_{u\to\infty} u^{-1}(x_\epsilon - L)x_\epsilon\|f^*\|^{\alpha}_{L^\alpha(\mathbb{R},\mathcal{B},\lambda)} \\
&= \frac{\epsilon}{(\alpha-1)\mu}.
\end{aligned} \tag{3.26}$$

Then letting $\epsilon \downarrow 0$, we see that

$$\psi_{12}^{(+)}(u) = o\left(u^{-(\alpha-1)}\right) \quad \text{as } u \to \infty, \tag{3.27}$$

and hence, combining (3.25),(3.27), we have

$$\psi_{1}^{(+)}(u) = o\left(u^{-(\alpha-1)}\right) \quad \text{as } u \to \infty. \tag{3.28}$$

Next pick $M > R$ and proceed by the following lemma which is analogous to Lemma 2.4:

Lemma 3.4 *For $x > M$,*

$$\sup_{0\le t<x-M}\frac{\int_0^t f^*(x-s)\,ds}{u+t\mu} \le \frac{\int_0^{x-M} f^*(x-s)\,ds}{u+(x-M)\mu}. \tag{3.29}$$

PROOF. For $x > M$ and any $t_0 \in [0, x-M)$, by the monotonicity of f^* on $[R,\infty)$ we have

$$
\begin{aligned}
\int_0^{t_0} f^*(x-s)\,ds - \frac{u+t_0\mu}{u+(x-M)\mu}\int_0^{x-M} f^*(x-s)\,ds
&= \left[1-\frac{u+t_0\mu}{u+(x-M)\mu}\right]\int_0^{t_0} f^*(x-s)\,ds \\
&\qquad - \frac{u+t_0\mu}{u+(x-M)\mu}\int_{t_0}^{x-M} f^*(x-s)\,ds \\
&\le \frac{(x-M-t_0)\mu}{u+(x-M)\mu}\, t_0 f^*(x-t_0) \\
&\qquad - \frac{u+t_0\mu}{u+(x-M)\mu}(x-M-t_0)f^*(x-t_0) \\
&= -\frac{u(x-M-t_0)f^*(x-t_0)}{u+(x-M)\mu} \le 0,
\end{aligned} \tag{3.30}
$$

and hence

$$
\frac{1}{u+t_0\mu}\int_0^{t_0} f^*(x-s)\,ds \le \frac{1}{u+(x-M)\mu}\int_0^{x-M} f(x-s)\,ds, \tag{3.31}
$$

yielding the desired result. □

Now write

$$
\begin{aligned}
\psi_3^{(+)}(u) \le \int_R^M \sup_{t\ge 0}\left(\frac{\int_0^t f(x-s)\,ds}{u+t\mu}\right)^\alpha dx + \int_M^\infty \sup_{0\le t<x-M}\left(\frac{\int_0^t f(x-s)\,ds}{u+t\mu}\right)^\alpha dx \\
+ \int_M^\infty \sup_{t\ge x-M}\left(\frac{\int_0^t f(x-s)\,ds}{u+t\mu}\right)^\alpha dx \\
:= \psi_{31}^{(+)}(u) + \psi_{32}^{(+)}(u) + \psi_{33}^{(+)}(u)
\end{aligned} \tag{3.32}
$$

First note that

$$
\psi_{31}^{(+)}(u) \le u^{-\alpha}(M-R)\|f\|^\alpha_{L^1(\mathbb{R},\mathcal{B},\lambda)} \tag{3.33}
$$

Additionally, it immediately follows from Lemma 3.4 that

$$
\psi_{32}^{(+)}(u) \le \int_M^\infty \left(\frac{\int_0^{x-M} f^*(x-s)\,ds}{u+(x-M)\mu}\right)^\alpha dx \le \left(\int_M^\infty f^*(s)ds\right)^\alpha \int_0^\infty (u+x\mu)^{-\alpha}dx \tag{3.34}
$$

Furthermore, it is easy to see that

$$
\psi_{33}^{(+)}(u) \le \|f\|^\alpha_{L^1(\mathbb{R},\mathcal{B},\lambda)} \int_0^\infty (u+x\mu)^{-\alpha}dx. \tag{3.35}
$$

Thus it follows from (3.32)-(3.35) that

$$\limsup_{u\to\infty} u^{\alpha-1}\psi_3^{(+)}(u) \le \frac{1}{(\alpha-1)\mu}\left[\|f\|^\alpha_{L^1(\mathbb{R},\mathcal{B},\lambda)} + \left(\int_M^\infty f^*(s)ds\right)^\alpha\right]. \tag{3.36}$$

Letting $M \uparrow \infty$ and recalling that $f^* \in L^1(\mathbb{R},\mathcal{B},\lambda)$, we have

$$\limsup_{u\to\infty} u^{\alpha-1}\psi_3^{(+)}(u) \le \frac{\|f\|^\alpha_{L^1(\mathbb{R},\mathcal{B},\lambda)}}{(\alpha-1)\mu}. \tag{3.37}$$

Finally, (3.28), (3.22), and (3.37) give (3.20), which in turn, yields (3.17).

Lastly observe that since $f \in L^1(\mathbb{R},\mathcal{B},\lambda)$, there exists $K > 0$ such that

$$\max\left(\int_{-\infty}^{-K} f(y)\,dy, \int_K^\infty f(y)\,dy\right) < 1.$$

Consequently,

$$\begin{aligned}\|h_t(\cdot)\|^\alpha_{L^\alpha(\mathbb{R},\mathcal{B},\lambda)} &= \int_{-\infty}^{-K} h_t^\alpha(x)\,dx + \int_{-K}^{K+t} h_t^\alpha(x)\,dx + \int_{K+t}^\infty h_t^\alpha(x)\,dx \\ &\le \int_{-\infty}^{-K} h_t(x)\,dx + \int_{-K}^{K+t} h_t^\alpha(x)\,dx + \int_{K+t}^\infty h_t(x)\,dx \\ &\le \int_{\mathbb{R}} h_t(x)\,dx + \int_{-K}^{K+t} h_t^\alpha(x)\,dx \\ &\le t\|f\|_{L^1(\mathbb{R},\mathcal{B},\lambda)} + (2K+t)\|f\|^\alpha_{L^1(\mathbb{R},\mathcal{B},\lambda)}.\end{aligned} \tag{3.38}$$

In particular,

$$\|h_t(\cdot)\|_{L^\alpha(\mathbb{R},\mathcal{B},\lambda)} = O(t^{1/\alpha}) \quad \text{as } t\to\infty,$$

and the desired result follows from Proposition 3.1. □

References

Alparslan, Uğur Tuncay & Gennady Samorodnitsky (2006), Ruin probability with stationary stable claims generated by conservative flows. Submitted to *Advances in Applied Probability*.

Baccelli, François & Pierre Brémaud (2003), *Elements of queueing theory*, Vol. 26 of *Applications of Mathematics (New York)*, second edn, Springer-Verlag, Berlin. Palm martingale calculus and stochastic recurrences, Stochastic Modelling and Applied Probability.

Braverman, Michael (2000), Exceedance probabilities of random sums with stable steps.

Braverman, Michael (2004), 'Tail probabilities of subadditive functionals on stable processes with continuous and discrete time', *Stochastic Process. Appl.* **112**(1), 157–183.

Brockwell, P. J., S. I. Resnick & R. L. Tweedie (1982), 'Storage processes with general release rule and additive inputs', *Adv. in Appl. Probab.* **14**(2), 392–433.

Embrechts, P. & N. Veraverbeke (1982), 'Estimates for the probability of ruin with special emphasis on the possibility of large claims', *Insurance Math. Econom.* **1**(1), 55–72.

Embrechts, Paul, Claudia Klüppelberg & Thomas Mikosch (1997), *Modelling extremal events*, Vol. 33 of *Applications of Mathematics (New York)*, Springer-Verlag, Berlin. For insurance and finance.

Harrison, J. Michael & Sidney I. Resnick (1976), 'The stationary distribution and first exit probabilities of a storage process with general release rule', *Math. Oper. Res.* **1**(4), 347–358.

Krengel, Ulrich (1985), *Ergodic theorems*, Vol. 6 of *de Gruyter Studies in Mathematics*, Walter de Gruyter & Co., Berlin. With a supplement by Antoine Brunel.

Mikosch, Thomas & Gennady Samorodnitsky (2000), 'Ruin probability with claims modeled by a stationary ergodic stable process', *Ann. Probab.* **28**(4), 1814–1851.

Rosiński, Jan (1995), 'On the structure of stationary stable processes', *Ann. Probab.* **23**(3), 1163–1187.

Samorodnitsky, Gennady & Murad S. Taqqu (1994), *Stable non-Gaussian random processes*, Stochastic Modeling, Chapman & Hall, New York. Stochastic models with infinite variance.