Applied Probability Trust (29 December 2006)

RUIN PROBABILITY WITH CERTAIN STATIONARY STABLE CLAIMS GENERATED BY CONSERVATIVE FLOWS

UĞUR TUNCAY ALPARSLAN,* *University of Nevada, Reno*

GENNADY SAMORODNITSKY,** *Cornell University*

Abstract

We study the ruin probability where the claim sizes are modeled by a stationary ergodic symmetric α-stable process. We exploit the flow representation of such processes, and we consider the processes generated by conservative flows. We focus on two classes of conservative α-stable processes (one discrete-time, and one continuous-time), and give results for the order of magnitude of the ruin probability as the initial capital goes to infinity. We also prove a solidarity property for null-recurrent Markov chains as an auxiliary result, which might be of independent interest.

Keywords: ruin probability; stable processes; heavy tails; ergodic theory; long range dependence; conservative flows; null recurrent Markov chains; fractional Brownian motion

2000 Mathematics Subject Classification: Primary 60G52

Secondary 62P05

1. Introduction

One of the popular problems of applied probability involves analyzing the exceedance probability of a threshold u given by

$$\psi(u) = P\left(\sup_{t\in\mathbb{T}}(S(t) - \mu(t)) > u\right), \tag{1.1}$$

* Postal address: Department of Mathematics and Statistics, University of Nevada, Reno, Nevada 89557

** Postal address: School of Operations Research and Industrial Engineering, Cornell University, Ithaca, New York 14853

where $\mathbf{S} = \{S(t),\ t \in \mathbb{T}\}$ is a random walk with index set $\mathbb{T}$, and $\underline{\mu} = \{\mu(t),\ t \in \mathbb{T}\}$ is a non-random drift term. This quantity has various interpretations in several different fields. In the context of risk theory and insurance, $\mathbf{S}$ can be considered as the cumulative claim size process, whereas $\underline{\mu}$ can be viewed as cumulative premium income on the insurance policy. In this case, one can view the exceedance probability as the *ruin probability with initial capital* u, or as the *ruin probability*, for short. (See [6].)

In this study we adhere to the language of insurance, however casually, although the results can be easily interpreted in other fields, including (but not limited to) queueing, and storage/dam models.

The research on ruin probabilities, in the sense of modern actuarial science, was mainly initiated in Sweden in the first half of the 20^{th} century. The foundations of the theory was laid down by Filip Lundberg in his Uppsala thesis (see [11]), while first mathematically substantial results appeared in a series of papers by Lundberg and Harald Cramér. The basic model coming out of these first contributions is widely referred to as the *Cramér-Lundberg model* (for details see, for instance, [6]). Since then there has been numerous extensions of the classical Cramér-Lundberg model with independent, identically distributed, light tailed claim sizes. More recently however, work in this area has turned to the more realistic setting of dependent claims. Moreover, empirical evidence in fields including insurance and financial markets, and the effort by banks, insurance companies, and governmental institutions to control risk associated with extreme events resulting in “large claims” has led to the theoretical interest in modeling “heavy tailed” phenomena.

In addition, from a theoretical point, the case of heavy-tailed, dependent claims is also interesting as it raises the question of the possibility of relating the dependence structure of the heavy-tailed stationary processes underlying the claims to the asymptotic behavior of the ruin probability. This becomes particularly challenging when the second moment of the claim sizes is infinite, so that it is not possible to use covariances to quantify the strength and the range of dependence.

In this study we focus on claim sizes modeled by stationary ergodic symmetric α-stable (SαS) processes, an important class of heavy tailed processes. We choose to work with $\alpha \in (1, 2)$, for which the claim process has a finite first moment but infinite second moment, and the ruin probability with a linear premium process is non-trivial. This, together with the fact that the probabilistic structure of these processes is relatively well understood, allows us to focus on the underlying dependence structure in the presence of heavy-tails.

The setup of SαS claims with $\mathbb{T} = \mathbb{Z}_+$, deterministic claim arrival processes, and constant premium rates has been addressed in [12], which is the origin of our current work. Based on the results of [8], the authors have observed that the order of magnitude of $\psi(u)$ for this model is $u^{-(\alpha-1)}$ in the case of iid claim sizes. Therefore, this is the "fastest" rate one can expect the ruin probability to decay in such a model. It is also shown *ibid* that for certain claim processes $\psi(u)$ decays as fast as $u^{-(\alpha-1)}$ even when the claim sizes are dependent. In the tradition of Mikosch and Samorodnitsky, we think of claim processes in this class as *short-range dependent.* They also show that for certain classes of SαS claims, $\psi(u)$ may decay slower than $u^{-(\alpha-1)}$. We think of these processes as *long-range dependent.*

In this study, we also investigate the case of $\mathbb{T} = \mathbb{R}_+$ utilizing recent results of [4].

Let now our claim process, $\mathbf{X} = \{X(t),\ t \in \mathbb{T}\}$, be a measurable stationary ergodic SαS process with $\alpha \in (1, 2)$ given in the form

$$X(t) = \int_E f_t(x) M(dx), \quad t \in \mathbb{T}, \tag{1.2}$$

where M is a SαS random measure on a measurable space $(E, \mathcal{E})$ with a $\sigma-$finite control measure m on $\mathcal{E}$, (i.e. M is an independently scattered random measure on $\mathcal{E}$ such that

$$E \exp\{i\lambda M(A)\} = \exp\{-|\lambda|^\alpha m(A)\}, \quad \lambda \in \mathbb{R},$$

for every $A \in \mathcal{E}$ with $m(A) < \infty$,) and $\{f_t\}_{t\in\mathbb{T}} \subset L^\alpha(E, \mathcal{E}, m)$. (See Section 3.3 of [19].)

Since we consider *stationary* SαS processes we can choose f_t to be in a particularly descriptive form given by

$$f_t(x) = a_t(x) \left[\frac{dm \circ \phi_t}{dm}(x)\right]^\alpha f \circ \phi_t(x), \quad x \in E,\ t \in \mathbb{T}, \tag{1.3}$$

where $\{\phi_t\}_{t\in\mathbb{T}}$ is a *non-singular* flow, (recall that a *flow* is a family of measurable maps from E onto E such that $\phi_{t_1+t_2} = \phi_{t_1} \circ \phi_{t_2}$ for all $t_1, t_2 \in \mathbb{T}$, and ϕ_0 is the identity function on E), $\{a_t\}_{t\in\mathbb{T}}$ is a *cocycle* for this flow (i.e. for every $t_1, t_2 \in \mathbb{T}$, $a_{t_1+t_2}(x) = a_{t_2}(x)a_{t_1} \circ \phi_{t_2}(x)$ for $m-$a.a. $x \in E$) taking values in $\{-1, 1\}$, and $f \in L^\alpha(E, \mathcal{E}, m)$. (See [16].)

This representation is particularly important as it brings up the possibility of relating the properties of a stationary SαS process to those of a flow and a single kernel. For instance, *Hopf decomposition* (see, e.g. [10],) of the flow $\{\phi_t\}_{t\in T}$ immediately implies that a stationary SαS process, $\mathbf{X}$, can be written (in distribution) as a sum of two independent stationary SαS processes

$$\mathbf{X} = \mathbf{X^D} + \mathbf{X^C}, \tag{1.4}$$

where $\mathbf{X^D}$ is given by representations (1.2) and (1.3) with a dissipative flow, and $\mathbf{X^C}$ is given by representations (1.2) and (1.3) with a conservative flow.

In this paper we investigate the asymptotic behavior of the ruin probability when the claims constitute a stationary SαS process generated purely by conservative flows, i.e. processes of the form $\mathbf{X^C}$ given in (1.4).

The case of stationary SαS claims of the form $\mathbf{X^D}$ is analyzed in a separate study and the results are presented in [2].

Intuitively, one expects the range of dependence of a stationary SαS process generated by a conservative flow to be longer than that of a stationary SαS process generated by a dissipative flow. Although a complete theory of risk processes with claims associated with conservative flows is lacking at the time of this study, and in general construction of processes generated by conservative flows is not effortless, factual support for such an intuition is provided by an example investigated in [12]. In their paper authors observe a class of conservative SαS processes constructed through a null-recurrent Markov chain (see [17] for details), and examine the asymptotic behavior of the ruin probability in a setting where the claims are modeled as a special case of this class and the premium process is a deterministic linear drift. Their results show that the ruin probability $\psi(u)$ in this case may decay much slower than $u^{-(\alpha-1)}$ even when the kernel in the integral representation (1.2) is "nice", i.e. in the context of ruin probabilities, at least the class of processes associated with conservative flows investigated in their example may be long-range dependent regardless of the kernel.

This is indeed a significant observation as the results given [2] suggest that in the risk theory context, for claims generated by dissipative flows, kernel in the integral representation of the claim process is the key factor in determining the range of dependence for the process.

In section 2 of this paper we focus on a related, but more general class of SαS processes constructed in [17], and studied in [12]. Our main result, which shows that the order of magnitude of the ruin probability $\psi(u)$ in the setting we describe below is $u^{-\gamma(\alpha-1)}L(u)$, where $L(\cdot)$ is a slowly varying function and $\gamma \in (0,1)$, is a generalization of the result given in [12]. We also prove a solidarity property for null-recurrent Markov chains as a subsidiary result, which might be of independent interest.

In section 3, we study the ruin probability in continuous time. In particular, we concentrate on a class of stationary SαS processes associated with conservative flows constructed using a fractional Brownian motion in [18]. We use a Brownian motion to construct our claim process and we show that in this setting the order of magnitude of the ruin probability $\psi(u)$ is given by $u^{-(\alpha-1)/2}$. We also conjecture that for a claim process associated with a fractional Brownian motion with self-similarity exponent $H \in (0,1)$, the order of magnitude is $u^{-H(\alpha-1)}$.

2. A discrete time claim process associated with a conservative flow

2.1. Setup and assumptions

Consider an irreducible, null-recurrent Markov chain, $\mathbf{Y} = \{Y_n,\ n \geq 1\}$, on $\mathbb{Z}$ with law $P_s(\cdot)$ on

$$E = \{\mathbf{y} = (y_0, y_1, y_2, \ldots):\ y_i \in \mathbb{Z}, i = 0, 1, 2, \ldots\}$$

corresponding to the initial state $y_0 = s \in \mathbb{Z}$.

Let $\pi = \{\pi_s,\ s \in \mathbb{Z}\}$ be the $\sigma-$finite invariant measure corresponding to the family $\{P_s,\ s \in \mathbb{Z}\}$ satisfying $\pi_0 = 1$, and define a $\sigma-$finite measure on the cylindrical $\sigma-$field of E by

$$m(\cdot) = \sum_{i=-\infty}^{\infty} \pi_i P_i(\cdot). \tag{2.1}$$

Note that this measure is invariant under the shift operator $\theta : E \to E$;

$$\theta(\mathbf{y}) = (y_1, y_2, \ldots), \quad \mathbf{y} = (y_0, y_1, y_2, \ldots) \in E.$$

We will model the claim size process, $\mathbf{X} = \{X_n,\ n \geq 1\}$, with a S$\alpha$S process defined by

$$X_n = \int_E f_n(\mathbf{y}) M(d\mathbf{y}), \quad \mathbf{y} \in E,\ \ n = 1, 2, 3, \ldots, \tag{2.2}$$

where M is a SαS random measure on E with control measure m given in (2.1), kernels f_n are given by

$$f_n(\mathbf{y}) = \sum_{s \in A} a_s 1_{[y_n = s]}, \quad n \geq 1,\ \ \mathbf{y} = (y_0, y_1, y_2, \ldots) \in E,$$

$A \subset \mathbb{Z}$ is a finite set, and $\{a_s,\ s \in A\}$ are positive reals. To avoid triviality assume $A \neq \emptyset$.

It follows from [17] that the process $\mathbf{X}$ given by the stochastic integral representation (2.2) is a stationary mixing process, and in particular is ergodic, and furthermore $\mathbf{X}$ is associated with a conservative flow.

For a given $\mathbf{y} \in E$ and $s \in \mathbb{Z}$, define the number of steps until the chain returns to state s for the first time as

$$\tau_s = \tau_s(\mathbf{y}) := \inf\{n \geq 1 :\ y_n = s\}.$$

Note that by the null-recurrence of the Markov chain $E_s \tau_s = \infty$, for any $s \in \mathbb{Z}$. We will further assume that there is a constant $\gamma \in (0, 1)$ and a slowly varying function L such that

$$P_0(\tau_0 \geq n) = n^{\gamma - 1} L(n). \tag{2.3}$$

For an integer s and a given $\mathbf{y} \in E$, define the number of visits to state s in n steps to be

$$N_n^{(s)} = N_n^{(s)}(\mathbf{y}) := \sum_{j=1}^{n} 1_{[y_j = s]}(\mathbf{y}),$$

and define

$$\eta_n^{(s)} := N_n^{(s)} n^{\gamma - 1} L(n), \quad s \in \mathbb{Z}.$$

Also for $\mathbf{y} \in E$, $s_0, s_1 \in \mathbb{Z}$, and $m \geq 1$, define the time spent in state s_1 between the $(m-1)^{st}$ and m^{th} visits to state s_0 as

$$W_m^{(s_0,s_1)} = W_m^{(s_0,s_1)}(\mathbf{y}) := \begin{cases} \sum_{j=\tau_{s_0}^{(m-1)}}^{\tau_{s_0}^{(m)}-1} 1_{[y_j=s_1]}, & \tau_{s_0}^{(m-1)} < \infty \\ 0, & \tau_{s_0}^{(m-1)} = \infty, \end{cases}$$

(here, for $s \in \mathbb{Z}$, $\tau_s^{(m)}$ is the time of m^{th} visit to state s with $\tau_s^{(0)} = 0$). Note that since we are considering a recurrent Markov chain, for any $m \geq 1$,

$$P_{s_0}(\tau_{s_0}^{(m-1)} = \infty) = 0,$$

and under P_{s_0}, $\left\{W_m^{(s_0,s_1)},\ m \geq 1\right\}$ are iid. Further notice

$$E_{s_0} W_m^{(s_0,s_1)} = \pi_{s_1}/\pi_{s_0}, \quad m \geq 1,$$

(see for instance Proposition 2.12.2 in [14]).

Finally, for a constant premium rate $\mu > 0$, let the cumulative premium process be given by

$$\underline{\mu} = \{\mu_n = n\mu,\ n \geq 1\},$$

and define the accumulated claim process $\mathbf{S} = \{S_n,\ n \geq 1\}$ by

$$S_0 = 0; \quad S_n = \sum_{i=1}^{n} X_n, \quad n = 1, 2, 3, \ldots.$$

Then the ruin probability given in (1.1) can be written as

$$\psi(u) = P\left(\sup_{n\geq 0}(S_n - \mu_n) > u\right), \quad u > 0. \tag{2.4}$$

2.2. A solidarity theorem for null recurrent Markov chains and the asymptotic analysis of the ruin probability

We start by giving a solidarity theorem regarding the tails of the return times to a state for a Markov chain with property (2.3). This result will be utilized throughout the remainder of this section, and it will be particularly important in determining the asymptotic behavior of the moments of the number of visits to a state given the initial state. Related solidarity theorems regarding the first moment of the number of visits to a state given the initial state has been given in [20]. However, Teugels's results on the first moments give the order of magnitude without calculating the exact multiplicative constant in the asymptotic form. Furthermore, his results regarding the transition probabilities assume that the slowly varying function given in (2.3) is monotone increasing. In this study we do not require this. Additionally, in our result below, we establish the exact asymptotic equivalence by specifying the multiplicative constant.

Theorem 2.1. *If (2.3) holds then for any* $s \in \mathbb{Z}$*,*

$$\pi_s P_s(\tau_s \geq n) \sim P_0(\tau_0 \geq n) \quad \text{as } n \to \infty.$$

Proof. For $s = 0$ the result holds trivially as $\pi_0 = 1$. Now fix $s \in \mathbb{Z} \setminus \{0\}$, and for any state $\tilde{s} \in \mathbb{Z}$ let

$$L_{n,\tilde{s}} := \tau_{\tilde{s}}^{\left(N_n^{(\tilde{s})}\right)}, \quad \text{and} \quad R_{n,\tilde{s}} := \tau_{\tilde{s}}^{\left(N_n^{(\tilde{s})}+1\right)}$$

be the time of the last visit to state $\tilde{s}$ before (or at) time n, and the time of the first visit to state $\tilde{s}$ after n, respectively.

Note that

$$\sum_{m=1}^{N_n^{(0)}} W_m^{(0,s)} = N_{L_{n,0}}^{(s)} \leq N_n^{(s)} \leq N_{R_{n,0}}^{(s)} = \sum_{m=1}^{N_n^{(0)}+1} W_m^{(0,s)}.$$

In particular,

$$\begin{aligned} \eta_n^{(0)} \left[\frac{1}{N_n^{(0)}} \sum_{m=1}^{N_n^{(0)}} W_m^{(0,s)} \right] &= n^{\gamma-1} L(n) N_{L_{n,0}}^{(s)} \\ &\leq \eta_n^{(s)} \\ &\leq n^{\gamma-1} L(n) N_{R_{n,0}}^{(s)} = \eta_n^{(0)} \left[\frac{1}{N_n^{(0)}} \sum_{m=1}^{N_n^{(0)}+1} W_m^{(0,s)} \right]. \end{aligned} \tag{2.5}$$

Next observe, for any two states $s_0, s_1 \in \mathbb{Z}$ it follows from Kolmogorov's strong law of large numbers that P_{s_0}-a.s.

$$\lim_{n\to\infty} \frac{1}{N_n^{(s_0)}} \sum_{m=1}^{N_n^{(s_0)}} W_m^{(s_0,s_1)} = \lim_{n\to\infty} \frac{1}{N_n^{(s_0)}} \sum_{m=1}^{N_n^{(s_0)}+1} W_m^{(s_0,s_1)} = E_{s_0} W_1^{(s_0,s_1)} = \frac{\pi_{s_1}}{\pi_{s_0}}. \tag{2.6}$$

Let $(Z_{1-\gamma})$ be a $(1-\gamma)$-stable subordinator, i.e. a positive increasing strictly $(1-\gamma)$-stable Lévy motion with

$$E\exp\{i\lambda Z_{1-\gamma}(1)\} = \exp\left\{-C_{1-\gamma}^{-1}|\lambda|^{1-\gamma}\left(1 - i\tan\frac{\pi(1-\gamma)}{2}\right)\right\}, \quad \lambda \in \mathbb{R},$$

and $C_{1-\gamma}$ is the usual constant associated with α-stable variables with $\alpha = 1-\gamma$. In other words, $Z_{1-\gamma}(1) \sim S_{1-\gamma}(\sigma_0, 1, 0)$, where $\sigma_0^{1-\gamma} = \Gamma(\gamma)\cos(\pi(1-\gamma)/2)$. In [12] it is shown that under P_0

$$\eta_n^{(0)} \Rightarrow Z_{1-\gamma}^{\gamma-1}(1).$$

Thus it follows from (2.5), (2.6), and Slutsky's theorem that

$$\frac{1}{\pi_s}\eta_n^{(s)} \Rightarrow Z_{1-\gamma}^{\gamma-1}(1) \tag{2.7}$$

under P_0.

We next show that (2.7) holds under P_s as well. Fix $x > 0$. Note that for sufficiently large n

$$P_0\left(\eta_n^{(s)} > x, \tau_s \geq n\right) \leq P_0\left(n^{\gamma-1}L(n) > x\right) = 0,$$

and hence it follows from the strong Markov property, that for n large,

$$\begin{aligned} P_0\left(\eta_n^{(s)} > x\right) &= P_0\left(\eta_n^{(s)} > x, \tau_s < n\right) \\ &\leq \sum_{i=1}^{n-1} P_0\left(n^{\gamma-1}L(n)N_n^{(s)} > x \middle|\, \tau_s = i\right) P_0(\tau_s = i) \\ &\leq P_s\left(\eta_n^{(s)} + n^{\gamma-1}L(n) > x\right) P_0(\tau_s < n) \\ &\leq P_s\left(\eta_n^{(s)} + n^{\gamma-1}L(n) > x\right). \end{aligned} \tag{2.8}$$

Therefore, we see that

$$\lim_{n\to\infty} P_0\left(\eta_n^{(s)} > x\right) \le \liminf_{n\to\infty} P_s\left(\eta_n^{(s)} > x\right). \tag{2.9}$$

Now let G_s^0 be the number of visits to state s before the first visit to 0. (Observe that G_s^0 has a geometric distribution under P_s.) Then for $x > 0$,

$$\begin{aligned} P_s\left(\eta_n^{(s)} > x\right) &= P_s\left(\eta_n^{(s)} > x, \tau_0 \ge n\right) + P_s\left(\eta_n^{(s)} > x, \tau_0 < n\right) \\ &\le P_s\left(\tau_0 \ge n\right) \\ &\quad + P_s\left[n^{\gamma-1}L(n)G_s^0 + n^{\gamma-1}L(n)\left(N_n^{(s)} - G_s^0\right) > x, \tau_0 < n\right]. \end{aligned} \tag{2.10}$$

Pick $\delta \in (0, 1-\gamma)$. Notice that as $n \to \infty$

$$n^{\delta+\gamma-1}L(n)G_s^0 \xrightarrow{P_s} 0.$$

Then as n tends to infinity

$$\begin{aligned} &P_s\left[n^{\gamma-1}L(n)G_s^0 + n^{\gamma-1}L(n)\left(N_n^{(s)} - G_s^0\right) > x, \tau_0 < n\right] \\ &\quad \le P_s\left(n^{\gamma-1}L(n)G_s^0 > n^{-\delta}\right) \\ &\qquad + P_s\left[n^{-\delta} + n^{\gamma-1}L(n)\left(N_n^{(s)} - G_s^0\right) > x, \tau_0 < n\right] \\ &\quad = P_s\left[n^{-\delta} + n^{\gamma-1}L(n)\left(N_n^{(s)} - G_s^0\right) > x, \tau_0 < n\right] + o(1). \end{aligned} \tag{2.11}$$

But by strong Markov property and Slutsky's theorem we have

$$\begin{aligned} &P_s\left[n^{-\delta} + n^{\gamma-1}L(n)\left(N_n^{(s)} - G_s^0\right) > x, \tau_0 < n\right] \\ &\qquad = \sum_{i=1}^{n-1} P_s\left[n^{-\delta} + n^{\gamma-1}L(n)\left(N_n^{(s)} - G_s^0\right) > x, \tau_0 = i\right] \\ &\qquad = \sum_{i=1}^{n-1} P_0\left(n^{-\delta} + n^{\gamma-1}L(n)N_{n-i}^{(s)} > x\right) P_s(\tau_0 = i) \\ &\qquad \le P_0\left(n^{-\delta} + n^{\gamma-1}L(n)N_n^{(s)} > x\right) P_s(\tau_0 < n) \\ &\qquad \le P_0\left(n^{-\delta} + \eta_n^{(s)} > x\right) \\ &\qquad \sim P_0\left(\eta_n^{(s)} > x\right). \end{aligned} \tag{2.12}$$

Combining (2.10)-(2.12) we have

$$P_s\left(\eta_n^{(s)} > x\right) \le P_0\left(\eta_n^{(s)} > x\right) + o(1). \tag{2.13}$$

It follows from (2.9) and (2.13) that (2.7) also holds under P_s.

Now define $\hat{a}_n := \inf\left\{k : \pi_s k^{1-\gamma} L^{-1}(k) \ge n\right\}$. Then for $y > 0$,

$$\begin{aligned} P_s\left(\frac{\tau_s^{(n)}}{\hat{a}_n} \le y\right) &= P_s\left(N_{[y\hat{a}_n]}^{(s)} \ge n\right) \\ &= P_s\left[\frac{1}{\pi_s}\eta_{[y\hat{a}_n]}^{(s)}\left(\frac{\pi_s \hat{a}_n^{1-\gamma} L^{-1}(\hat{a}_n)}{n}\right)\left(\frac{L(\hat{a}_n)}{L(y\hat{a}_n)}\right)\left(\frac{(y\hat{a}_n)^{\gamma-1} L(y\hat{a}_n)}{[y\hat{a}_n]^{\gamma-1} L([y\hat{a}_n])}\right) \ge y^{\gamma-1}\right]. \end{aligned} \tag{2.14}$$

By the slow variation of L,

$$\lim_{n\to\infty}\left[\left(\frac{\pi_s \hat{a}_n^{1-\gamma} L^{-1}(\hat{a}_n)}{n}\right)\left(\frac{L(\hat{a}_n)}{L(y\hat{a}_n)}\right)\left(\frac{(y\hat{a}_n)^{\gamma-1} L(y\hat{a}_n)}{[y\hat{a}_n]^{\gamma-1} L([y\hat{a}_n])}\right)\right] = 1.$$

Therefore, it follows from (2.7) holding under P_s, Slutsky's theorem, and the self-similarity of the stable subordinator that as n goes to infinity, for $y > 0$,

$$P_s\left(\frac{\tau_s^{(n)}}{\hat{a}_n} \le y\right) \sim P\left(Z_{1-\gamma}^{\gamma-1}(1) \ge y^{\gamma-1}\right) = P\left(Z_{1-\gamma}(1) \le y\right), \tag{2.15}$$

i.e. $\hat{a}_n^{-1}\tau_s^{(n)} \Rightarrow Z_{1-\gamma}(1)$ under P_s. Consequently, (see, for instance Theorem 1.8.1, p.50 of [19]),

$$P_s(\tau_s > x) = x^{\gamma-1}\hat{L}(x), \tag{2.16}$$

for a slowly varying function $\hat{L}$, and moreover

$$P_s(\tau_s > \hat{a}_n) \sim \frac{1}{n}, \quad n \to \infty. \tag{2.17}$$

Thus,

$$\hat{a}_n^{\gamma-1}\hat{L}(\hat{a}_n) \sim \frac{1}{n}, \quad n \to \infty. \tag{2.18}$$

Furthermore, defining $a_n := \inf\left\{k : k^{1-\gamma} L^{-1}(k) \ge n\right\}$, we immediately see that

$$a_n^{\gamma-1} L(a_n) \sim \frac{1}{n}, \quad n \to \infty. \tag{2.19}$$

In addition,

$$\lim_{n\to\infty} \frac{a_n}{\hat{a}_n} = \pi_s^{\frac{1}{1-\gamma}}. \tag{2.20}$$

Consequently, it follows from (2.18)-(2.20), and the fact that $\hat{L}$ is slowly varying that, as n tends to infinity

$$\hat{L}(a_n) \sim \hat{L}(\hat{a}_n) \sim \pi_s^{-1} L(a_n), \tag{2.21}$$

and so

$$\lim_{n\to\infty} \frac{\hat{L}(n)}{L(n)} = \lim_{n\to\infty} \frac{\hat{L}(a_{[n^{1-\gamma}L^{-1}(n)]})}{L(a_{[n^{1-\gamma}L^{-1}(n)]})} = \pi_s^{-1}, \tag{2.22}$$

which gives the desired result.

Define

$$\psi_0(u) = \frac{C_\alpha}{2} \int_{\mathbb{R}} \sup_{n\geq 0} \frac{(h_n(x))_+^\alpha}{(u+\mu_n)^\alpha}\, dx + \frac{C_\alpha}{2} \int_{\mathbb{R}} \sup_{n\geq 0} \frac{(-h_n(x))_+^\alpha}{(u+\mu_n)^\alpha}\, dx, \quad u > 0, \tag{2.23}$$

where

$$C_\alpha = \left(\int_0^\infty x^{-\alpha} \sin x \, dx \right)^{-1}. \tag{2.24}$$

The following two results can be established via Theorem 2.1 and an argument parallel to that in [12].

Proposition 2.1. *Given (2.3) the following relation holds:*

$$\psi(u) \sim \psi_0(u), \quad \textit{as } u \to \infty. \tag{2.25}$$

Lemma 2.1. *For $s \in \mathbb{Z}$,*

$$m(\tau_s = k) = \pi_s P_s(\tau_s \geq k), \quad k = 1, 2, \ldots, \tag{2.26}$$

and

$$m(\tau_s \leq n) \sim \gamma^{-1} n^\gamma L(n) \quad \textit{as } n \to \infty. \tag{2.27}$$

Next theorem establishes the main result of this section by showing that the ruin probability $\psi(u)$ may decay very slowly as the initial capital u increases in the setting described above. Note that, unlike Theorem 3.2 of [12], this result is only stated for

$\gamma \in (0,1)$, as the solidarity property proved in Theorem 2.1 was shown only for these values of γ. However, we expect the solidarity property to hold for $\gamma = 1$ as well, which in turn should make the following result extendable to this case.

Theorem 2.2. *Under the assumption (2.3) the following relation holds:*

$$\psi(u) \sim \left(\sum_{s\in A} a_s \pi_s\right)^{\alpha} A_{\alpha,\gamma}\mu^{\gamma(\alpha-1)-\alpha}u^{-\gamma(\alpha-1)}L^{-(\alpha-1)}(u) \quad \text{as } u \to \infty, \tag{2.28}$$

where

$$A_{\alpha,\gamma} = \frac{C_\alpha \beta(\gamma, \gamma(\alpha-1))}{2} E\left(\sup_{t\geq 1}\frac{t-1}{Z_{1-\gamma}(t)}\right)^{\alpha(1-\gamma)},$$

and $\beta(\cdot,\cdot)$ is the beta function.

Proof. In the light of Proposition 2.1 it is enough to show the result for $\psi_0(u)$. Start by fixing $s_0 \in A$:

Lemma 2.2. *The following relation holds:*

$$\begin{aligned} g(u) &:= E_{s_0}\left[\sup_{n\geq 0}\left(\frac{\sum_{s\in A} a_s N_n^{(s)}}{u+n}\right)^{\alpha}\right] \\ &\sim \left(\sum_{s\in A} a_s\pi_s\right)^{\alpha} E\left(\sup_{t\geq 1}\frac{t-1}{Z_{1-\gamma}(t)}\right)^{\alpha(1-\gamma)} u^{-\gamma\alpha}L^{-\alpha}(u), \end{aligned} \tag{2.29}$$

as $u \to \infty$.

Proof. It is easy to see by (2.6), and the argument given in (2.5) that

$$\lim_{n\to\infty}\sum_{s\in A}\frac{a_s N_n^{(s)}}{N_n^{(s_0)}} = \sum_{s\in A}\frac{a_s\pi_s}{\pi_{s_0}} \quad P_{s_0} - \text{a.s.} \tag{2.30}$$

Also note that as shown previously

$$\frac{1}{\pi_{s_0}}\eta_n^{(s_0)} \Rightarrow Z_{1-\gamma}^{\gamma-1}(1) \quad \text{under} \quad P_{s_0}. \tag{2.31}$$

Therefore, now Slutsky's theorem gives

$$\sum_{s\in A} a_s\eta_n^{(s)} = \eta_n^{(s_0)}\left[\sum_{s\in A}\frac{a_s N_n^{(s)}}{N_n^{(s_0)}}\right] \Rightarrow \left(\sum_{s\in A} a_s\pi_s\right) Z_{1-\gamma}^{\gamma-1}(1) \quad \text{under} \quad P_{s_0}. \tag{2.32}$$

Moreover, by Theorem 2.1 and an argument similar to that of Proposition 3.4 of [13] one can show that all power moments of $\eta_n^{(s)}$ converge under P_s. In particular, this together with continuous mapping theorem imply, for any $\delta > 0$, $\left\{\left(\eta_n^{(s)}\right)^{\alpha+\delta}\right\}_{n\geq 1}$ are uniformly integrable under P_s. Thus, it follows from Theorem 6.5.1 of [15] that

$$\sup_{n\geq 1} E_s\left[\left(\eta_n^{(s)}\right)^{\alpha+\delta}\right] < \infty.$$

Next, for any $s \neq s_0$, observe that by strong Markov property, and Hölder's inequality,

$$\begin{aligned}
\sup_{n\geq 1} E_{s_0}\left[\left(\eta_n^{(s)}\right)^{\alpha+\delta}\right] &= \sup_{n\geq 1} E_{s_0}\left\{\left(n^{\gamma-1}L(n)\right)^{\alpha+\delta} E_{s_0}\left[\left(N_n^{(s)}\right)^{\alpha+\delta}\middle|\tau_s\right]\right\} \\
&= \sup_{n\geq 1} E_{s_0}\left\{\left(n^{\gamma-1}L(n)\right)^{\alpha+\delta} E_s\left[\left(1+N_{n-\tau_s}^{(s)}\right)^{\alpha+\delta}\right]\right\} \\
&\leq 2^{\alpha+\delta-1}\sup_{n\geq 1}\left\{\left(n^{\gamma-1}L(n)\right)^{\alpha+\delta} + E_s\left[\left(n^{\gamma-1}L(n)\right)^{\alpha+\delta}\left(N_{n-\tau_s}^{(s)}\right)^{\alpha+\delta}\right]\right\} \\
&\leq 2^{\alpha+\delta-1}\left\{1+\sup_{n\geq 1} E_s\left[\left(\eta_n^{(s)}\right)^{\alpha+\delta}\right]\right\} < \infty.
\end{aligned} \tag{2.33}$$

So the "crystal ball condition" (see for example p.184 of [15]) is satisfied and hence we conclude that $\left\{\left(\eta_n^{(s)}\right)^{\alpha}\right\}_{n\geq 1}$ are uniformly integrable under P_{s_0}. This, together with the fact that $\left\{\left(\eta_n^{(s_0)}\right)^{\alpha}\right\}_{n\geq 1}$ are uniformly integrable under P_{s_0} implies that $\left\{\left(\sum_{s\in A} a_s\eta_n^{(s)}\right)^{\alpha}\right\}_{n\geq 1}$ are uniformly integrable under P_{s_0} as

$$\left(\sum_{s\in A} a_s\eta_n^{(s)}\right)^{\alpha} \leq [\#(A)]^{\alpha-1}\sum_{s\in A} a_s^{\alpha}\left(\eta_n^{(s)}\right)^{\alpha}.$$

Then recalling (2.32), and using continuous mapping theorem we see that

$$\lim_{n\to\infty} E_{s_0}\left(\sum_{s\in A} a_s\eta_n^{(s)}\right)^{\alpha} = \left(\sum_{s\in A} a_s\pi_s\right)^{\alpha} EZ_{1-\gamma}^{\alpha(\gamma-1)}(1).$$

In particular, we have

$$E_{s_0}\left(\sum_{s\in A} a_s N_n^{(s)}\right)^{\alpha} \sim \left(\sum_{s\in A} a_s\pi_s\right)^{\alpha} n^{\alpha(1-\gamma)}L^{-\alpha}(n)EZ_{1-\gamma}^{-\alpha(1-\gamma)}(1), \tag{2.34}$$

as $n \to \infty$.

Now for any $K > 0$ consider

$$g_K(u) := E_{s_0}\left[\sup_{0\le n\le uK}\left(\frac{\sum_{s\in A} a_s N_n^{(s)}}{u+n}\right)^{\alpha}\right],$$

and

$$g^K(u) := E_{s_0}\left[\sup_{n>uK}\left(\frac{\sum_{s\in A} a_s N_n^{(s)}}{u+n}\right)^{\alpha}\right].$$

An argument argument similar to that given in Lemma 3.4 of [12] yields

$$\lim_{K\uparrow\infty}\limsup_{u\to\infty} u^{\alpha\gamma}L^{\alpha}(u)g^K(u) = 0. \tag{2.35}$$

We will next bound $g_K(u)$. First, notice it is shown in [12] that as u tends to infinity,

$$\sup_{0\le n\le uK}\frac{u^\gamma L(u)N_n^{(0)}}{(u+n)} \Rightarrow \sup_{1\le t\le K+1}\left(\frac{t-1}{Z_{1-\gamma}(t)}\right)^{1-\gamma}, \quad \text{under } P_0.$$

One can use the same argument and Theorem 2.1 to easily see that

$$\sup_{0\le n\le uK}\frac{u^\gamma L(u)N_n^{(s_0)}}{\pi_{s_0}(u+n)} \Rightarrow \sup_{1\le t\le K+1}\left(\frac{t-1}{Z_{1-\gamma}(t)}\right)^{1-\gamma}, \quad \text{under } P_{s_0}. \tag{2.36}$$

Next observe that for $m \ge 1$,

$$\begin{aligned}
&\sup_{0\le n\le uK}\frac{u^\gamma L(u)\sum\limits_{s\in A} a_s N_n^{(s)}}{(u+n)} \\
&\quad\le \sum_{n=0}^{m-1}\frac{u^\gamma L(u)\sum_{s\in A} a_s N_n^{(s)}}{(u+n)} + \sup_{m\le n\le uK}\frac{u^\gamma L(u)\sum_{s\in A} a_s N_n^{(s)}}{(u+n)} \\
&\quad\le m^2\sum_{s\in A} a_s u^{\gamma-1}L(u) + \sup_{m\le n\le uK}\left[\frac{u^\gamma L(u)N_n^{(s_0)}}{(u+n)}\left(\sum_{s\in A}\frac{a_s N_n^{(s)}}{N_n^{(s_0)}}\right)\right] \\
&\quad\le m^2\sum_{s\in A} a_s u^{\gamma-1}L(u) + \left[\sup_{m\le n}\frac{\sum\limits_{s\in A} a_s N_n^{(s)}}{N_n^{(s_0)}}\right]\left[\sup_{0\le n\le uK}\frac{u^\gamma L(u)N_n^{(s_0)}}{(u+n)}\right].
\end{aligned} \tag{2.37}$$

Furthermore, for $\varepsilon \in (0,1)$ define

$$\mathrm{T}_\varepsilon := \inf \left\{ k \geq 1 : \sum_{s \in A} \frac{a_s N_n^{(s)}}{N_n^{(s_0)}} \geq (1-\varepsilon) \sum_{s \in A} \frac{a_s \pi_s}{\pi_{s_0}},\ n \geq k \right\}.$$

It follows from (2.30) that T_ε is finite P_{s_0}-a.s. Then,

$$\begin{aligned} \sup_{0 \leq n \leq uK} \frac{u^\gamma L(u) \sum_{s \in A} a_s N_n^{(s)}}{(u+n)} &\geq \sup_{\mathrm{T}_\varepsilon \leq n \leq uK} \left[\frac{u^\gamma L(u) N_n^{(s_0)}}{(u+n)} \left(\sum_{s \in A} \frac{a_s N_n^{(s)}}{N_n^{(s_0)}} \right) \right] \\ &\geq (1-\varepsilon) \sum_{s \in A} \frac{a_s \pi_s}{\pi_{s_0}} \left[\sup_{\mathrm{T}_\varepsilon \leq n \leq uK} \frac{u^\gamma L(u) N_n^{(s_0)}}{(u+n)} \right] \\ &\geq (1-\varepsilon) \sum_{s \in A} \frac{a_s \pi_s}{\pi_{s_0}} \left[\sup_{0 \leq n \leq uK} \frac{u^\gamma L(u) N_n^{(s_0)}}{(u+n)} - \sup_{0 \leq n \leq \mathrm{T}_\varepsilon} \frac{u^\gamma L(u) N_n^{(s_0)}}{(u+n)} \right] \\ &\geq (1-\varepsilon) \sum_{s \in A} \frac{a_s \pi_s}{\pi_{s_0}} \left[\sup_{0 \leq n \leq uK} \frac{u^\gamma L(u) N_n^{(s_0)}}{(u+n)} - u^{\gamma-1} L(u) \mathrm{T}_\varepsilon \right] \end{aligned} \tag{2.38}$$

Notice as u goes to infinity

$$u^{\gamma-1} L(u) \mathrm{T}_\varepsilon \overset{P_{s_0}}{\to} 0.$$

Now recalling (2.36) and Slutsky's theorem, then letting u go to infinity in (2.37) and (2.38), and finally letting m in (2.37) go to infinity and ε in (2.38) go to 0, we conclude that under P_{s_0}

$$\sup_{0 \leq n \leq uK} \frac{u^\gamma L(u) \sum_{s \in A} a_s N_n^{(s)}}{(u+n)} \Rightarrow \left(\sum_{s \in A} a_s \pi_s \right) \sup_{1 \leq t \leq K+1} \left(\frac{t-1}{Z_{1-\gamma}(t)} \right)^{1-\gamma}. \tag{2.39}$$

Moreover, notice that for any fixed $K > 0$,

$$\left(\sup_{0 \leq n \leq uK} \frac{u^\gamma L(u) \sum_{s \in A} a_s N_n^{(s)}}{(u+n)} \right)^\alpha \leq (constant) \left(\sum_{s \in A} a_s \eta_{[uK]}^{(s)} \right)^\alpha, \tag{2.40}$$

and the variables on the right hand side are uniformly integrable under P_{s_0} implying that

$$\left\{ \sup_{0 \leq n \leq uK} \left(\frac{u^\gamma L(u) \sum_{s \in A} a_s N_n^{(s)}}{u+n} \right)^\alpha \right\}_{u \geq 0}$$

are uniformly integrable under P_{s_0}. Thus, in particular, applying continuous mapping theorem we have

$$\begin{aligned}\lim_{u\to\infty} u^{\alpha\gamma}L^{\alpha}(u)g_K(u) &= \lim_{u\to\infty} E_{s_0}\left[\sup_{0\le n\le uK}\left(\frac{u^{\gamma}L(u)\sum_{s\in A}a_sN_n^{(s)}}{u+n}\right)^{\alpha}\right]\\ &= \left(\sum_{s\in A}a_s\pi_s\right)^{\alpha} E\left(\sup_{1\le t\le K+1}\frac{t-1}{Z_{1-\gamma}(t)}\right)^{\alpha(1-\gamma)}.\end{aligned} \tag{2.41}$$

In addition, it is shown in [12] that for any $p>0$

$$E\left(\sup_{t\ge1}\frac{t-1}{Z_{1-\gamma}(t)}\right)^{p}<\infty,$$

and hence letting K increase to infinity and recalling (2.35) we have

$$\lim_{u\to\infty} u^{\alpha\gamma}L^{\alpha}(u)g(u) = \left(\sum_{s\in A}a_s\pi_s\right)^{\alpha} E\left(\sup_{t\ge1}\frac{t-1}{Z_{1-\gamma}(t)}\right)^{\alpha(1-\gamma)}<\infty. \tag{2.42}$$

To proceed with the proof of the theorem notice that

$$\begin{aligned}\frac{2\psi_0(u)}{C_\alpha} &= \int_E \sup_{n\ge0}\frac{\left(\sum_{k=1}^n f_k(\mathbf{y})\right)_+^{\alpha}+\left(-\sum_{k=1}^n f_k(\mathbf{y})\right)_+^{\alpha}}{(u+n\mu)^{\alpha}}\,m(d\mathbf{y})\\ &= \sum_{i=-\infty}^{\infty}\pi_iE_i\left[\sup_{n\ge0}\left(\frac{\sum_{s\in A}a_sN_n^{(s)}}{u+n\mu}\right)^{\alpha}\right].\end{aligned} \tag{2.43}$$

For $A=\{s_0\}$, the desired result easily follows from strong Markov property, Lemma 2.1, Theorem 2.1, Lemma 2.2, and the proof of Theorem 3.2 of [12].

For $A\neq\{s_0\}$ write

$$\begin{aligned}\sum_{s\in A}a_sN_n^{(s)} &= \sum_{s\in A\setminus\{s_0\}}a_s\left(G_s^{s_0}1_{[\tau_s\le\tau_{s_0}\wedge n]}\wedge N_n^{(s)}\right)\\ &\quad+\left[\sum_{s\in A}a_sN_n^{(s)}-\sum_{s\in A\setminus\{s_0\}}a_s\left(G_s^{s_0}1_{[\tau_s\le\tau_{s_0}\wedge n]}\wedge N_n^{(s)}\right)\right],\end{aligned} \tag{2.44}$$

where for any states $s_1,s_2\in\mathbb{Z}$, and $\mathbf{y}\in E$,

$$G_{s_1}^{s_2}=G_{s_1}^{s_2}(\mathbf{y}):=\sum_{i=1}^{\tau_{s_2}(\mathbf{y})}1_{[y_i=s_1]},$$

i.e. $G_s^{s_0}$ is the number of visits to state s before the first visit to state s_0. (Note that $G_{s_1}^{s_2}$ has a geometric distribution under P_{s_1}.)

Now we collect some intermediate results, which will be combined at the last stage.

Observe that

$$\begin{aligned}\sum_{i=-\infty}^{\infty} \pi_i E_i &\left[\sup_{n\geq 0}\left(\frac{\sum_{s\in A\setminus\{s_0\}} a_s \left(G_s^{s_0} 1_{[\tau_s\leq\tau_{s_0}\wedge n]} \wedge N_n^{(s)}\right)}{u+n\mu}\right)^{\alpha}\right] \\ &\leq \sum_{i=-\infty}^{\infty} \pi_i E_i \left[\left(\sum_{s\in A\setminus\{s_0\}} \sup_{n\geq 0}\left(\frac{a_s G_s^{s_0} 1_{[\tau_s\leq\tau_{s_0}\wedge n]}}{u+n\mu}\right)\right)^{\alpha}\right] \\ &= \sum_{i=-\infty}^{\infty} \pi_i E_i \left[\left(\sum_{s\in A\setminus\{s_0\}} \frac{a_s G_s^{s_0}}{u+\tau_s\mu}\right)^{\alpha}\right],\end{aligned} \tag{2.45}$$

then it follows from Hölder's inequality and Fubini's theorem that

$$\begin{aligned}\sum_{i=-\infty}^{\infty} \pi_i E_i &\left[\left(\sum_{s\in A\setminus\{s_0\}} \frac{a_s G_s^{s_0}}{u+\tau_s\mu}\right)^{\alpha}\right] \\ &\leq [\#(A)]^{\alpha-1} \sum_{s\in A\setminus\{s_0\}} \sum_{i=-\infty}^{\infty} \pi_i E_i \left[\left(\frac{a_s G_s^{s_0}}{u+\tau_s\mu}\right)^{\alpha}\right],\end{aligned} \tag{2.46}$$

and by the strong Markov property,

$$\begin{aligned}\sum_{s\in A\setminus\{s_0\}} \sum_{i=-\infty}^{\infty} \pi_i E_i &\left[\left(\frac{a_s G_s^{s_0}}{u+\tau_s\mu}\right)^{\alpha}\right] \\ &= \sum_{s\in A\setminus\{s_0\}} a_s^{\alpha} \sum_{i=-\infty}^{\infty} \sum_{k=1}^{\infty} \pi_i E_i \left[\left(\frac{G_s^{s_0}}{u+\tau_s\mu}\right)^{\alpha} \middle| \tau_s = k\right] P_i(\tau_s = k) \\ &= \sum_{s\in A\setminus\{s_0\}} a_s^{\alpha} E_s\left[(G_s^{s_0}+1)^{\alpha}\right] \sum_{k=1}^{\infty} (u+k\mu)^{-\alpha} m(\tau_s = k).\end{aligned} \tag{2.47}$$

So by Lemma 2.1, as $u \to \infty$,

$$\begin{aligned}\sum_{i=-\infty}^{\infty} \pi_i E_i \left[\left(\sum_{s\in A\setminus\{s_0\}} \frac{a_s G_s^{s_0}}{u+\tau_s\mu}\right)^{\alpha}\right] &\leq (constant)(u+\mu)^{-(\alpha-1)} \\ &= o\left(u^{-\gamma(\alpha-1)} L^{-(\alpha-1)}(u)\right).\end{aligned} \tag{2.48}$$

Additionally, by the strong Markov property, Lemma 2.1, Theorem 2.1, Lemma 2.2, and Lemma 3.6 of [12],

$$
\begin{aligned}
&\sum_{i=-\infty}^{\infty} \pi_i E_i \left[\sup_{n \geq \tau_{s_0}} \left(\frac{\sum_{s\in A} a_s N_n^{(s)} - \sum_{s\in A\setminus\{s_0\}} a_s G_s^{s_0}}{u+n\mu} \right)^{\alpha} \right] \\
&\quad = \sum_{k=1}^{\infty} m(\tau_{s_0}=k) E_{s_0} \left[\sup_{n\geq 0} \left(\frac{\sum_{s\in A} a_s N_n^{(s)}}{u+(n+k)\mu} \right)^{\alpha} \right] \\
&\quad = \mu^{-\alpha} \sum_{k=1}^{\infty} P_0(\tau_0 \geq k) g(k+u/\mu) \\
&\quad \sim \frac{2\left(\sum_{s\in A} a_s \pi_s\right)^{\alpha} A_{\alpha,\gamma}}{C_\alpha} \mu^{\gamma(\alpha-1)-\alpha} u^{-\gamma(\alpha-1)} L^{-(\alpha-1)}(u) \quad \text{as } u\to\infty.
\end{aligned}
\tag{2.49}
$$

Furthermore, notice that $G_s^{s_0} = G_s^{s_0} 1_{[\tau_s<\tau_{s_0}]} \leq G_s^{s_0}(u+\tau_{s_0}\mu)/(u+\tau_s\mu)$. Consequently by (2.48) we have

$$
\sum_{i=-\infty}^{\infty} \pi_i E_i \left[\left(\frac{\sum\limits_{s\in A\setminus\{s_0\}} a_s G_s^{s_0}}{u+\tau_{s_0}\mu} \right)^{\alpha} \right] = o\left(u^{-\gamma(\alpha-1)} L^{-(\alpha-1)}(u) \right) \quad \text{as } u\to\infty. \tag{2.50}
$$

Now for any $M>0$,

$$
\begin{aligned}
&\sum_{i=-\infty}^{\infty} \pi_i E_i \left(\sup_{n\geq\tau_{s_0}} \frac{\left(\sum\limits_{s\in A\setminus\{s_0\}} a_s G_s^{s_0}\right)\left(\sum\limits_{s\in A} a_s N_n^{(s)} - \sum\limits_{s\in A\setminus\{s_0\}} a_s G_s^{s_0}\right)^{\alpha-1}}{(u+n\mu)^{\alpha}} \right) \\
&\quad \leq \sum_{i=-\infty}^{\infty} \pi_i E_i \left(\sup_{n\geq\tau_{s_0}} \frac{\left(\sum\limits_{s\in A\setminus\{s_0\}} a_s G_s^{s_0}\right) 1_{\left[\sum\limits_{s\in A} a_s N_n^{(s)} < (M+1) \sum\limits_{s\in A\setminus\{s_0\}} a_s G_s^{s_0}\right]}}{\left(\sum\limits_{s\in A} a_s N_n^{(s)}\right)^{1-\alpha} (u+n\mu)^{\alpha}} \right) \\
&\qquad + \frac{1}{M} \sum_{i=-\infty}^{\infty} \pi_i E_i \left(\sup_{n\geq\tau_{s_0}} \left(\frac{\sum\limits_{s\in A} a_s N_n^{(s)} - \sum\limits_{s\in A\setminus\{s_0\}} a_s G_s^{s_0}}{u+n\mu} \right)^{\alpha} \right)
\end{aligned}
\tag{2.51}
$$

and it follows from (2.49) and (2.50) that as $u \to \infty$,

$$\begin{aligned}
&\sum_{i=-\infty}^{\infty} \pi_i E_i \left(\sup_{n \geq \tau_{s_0}} \frac{\left(\sum_{s \in A \setminus \{s_0\}} a_s G_s^{s_0} \right) \left(\sum_{s \in A} a_s N_n^{(s)} - \sum_{s \in A \setminus \{s_0\}} a_s G_s^{s_0} \right)^{\alpha-1}}{(u+n\mu)^{\alpha}} \right) \\
&\leq (M+1)^{\alpha-1} \sum_{i=-\infty}^{\infty} \pi_i E_i \left[\left(\frac{\sum_{s \in A \setminus \{s_0\}} a_s G_s^{s_0}}{u + \tau_{s_0}\mu} \right)^{\alpha} \right] \\
&\quad + \frac{1}{M} \mu^{-\alpha} \sum_{k=1}^{\infty} P_0(\tau_0 \geq k) g(k + u/\mu) \\
&\sim M^{-1} \frac{2 \left(\sum_{s \in A} a_s \pi_s \right)^{\alpha} A_{\alpha,\gamma}}{C_\alpha} \mu^{\gamma(\alpha-1)-\alpha} u^{-\gamma(\alpha-1)} L^{-(\alpha-1)}(u).
\end{aligned} \tag{2.52}$$

But, since $M > 0$ is arbitrary we conclude that

$$\begin{aligned}
&\sum_{i=-\infty}^{\infty} \pi_i E_i \left(\sup_{n \geq \tau_{s_0}} \frac{\left(\sum_{s \in A \setminus \{s_0\}} a_s G_s^{s_0} \right) \left(\sum_{s \in A} a_s N_n^{(s)} - \sum_{s \in A \setminus \{s_0\}} a_s G_s^{s_0} \right)^{\alpha-1}}{(u+n\mu)^{\alpha}} \right) \\
&\quad = o\left(u^{-\gamma(\alpha-1)} L^{-(\alpha-1)}(u) \right) \quad \text{as } \ u \to \infty.
\end{aligned} \tag{2.53}$$

Lastly, observe that

$$\begin{aligned}
&n < \tau_{s_0} \implies \sum_{s \in A} a_s N_n^{(s)} - \sum_{s \in A \setminus \{s_0\}} a_s \left(G_s^{s_0} 1_{[\tau_s \leq \tau_{s_0} \wedge n]} \wedge N_n^{(s)} \right) = 0, \\
&n \geq \tau_{s_0} \implies G_s^{s_0} 1_{[\tau_s \leq \tau_{s_0} \wedge n]} \wedge N_n^{(s)} = G_s^{s_0},
\end{aligned} \tag{2.54}$$

and notice by the convexity of the function $c(x) = x^\alpha$ for $x \geq 0$, for any $x_0, y_0 \geq 0$, we have

$$\begin{aligned}
&\frac{(x_0+y_0)^\alpha - x_0^\alpha}{y_0} \leq \alpha (x_0+y_0)^{\alpha-1} \leq \alpha (x_0^{\alpha-1} + y_0^{\alpha-1}) \\
&\implies (x_0+y_0)^\alpha \leq x_0^\alpha + \alpha x_0^{\alpha-1} y_0 + \alpha y_0^\alpha.
\end{aligned} \tag{2.55}$$

So it follows from (2.43), (2.44), (2.54) and (2.55) that

$$
\begin{aligned}
\frac{2\psi_0(u)}{C_\alpha} \leq \sum_{i=-\infty}^{\infty} \pi_i E_i &\left[\sup_{n \geq \tau_{s_0}} \left(\frac{\sum_{s\in A} a_s N_n^{(s)} - \sum_{s\in A\setminus\{s_0\}} a_s G_s^{s_0}}{u+n\mu} \right)^{\alpha} \right] \\
+ \alpha \sum_{i=-\infty}^{\infty} \pi_i E_i &\left[\sup_{n \geq \tau_{s_0}} \frac{\left(\sum_{s\in A\setminus\{s_0\}} a_s G_s^{s_0}\right)\left(\sum_{s\in A} a_s N_n^{(s)} - \sum_{s\in A\setminus\{s_0\}} a_s G_s^{s_0}\right)^{\alpha-1}}{(u+n\mu)^\alpha} \right] \\
+ \alpha \sum_{i=-\infty}^{\infty} \pi_i E_i &\left[\sup_{n\geq 0} \left(\frac{\sum_{s\in A\setminus\{s_0\}} a_s \left(G_s^{s_0} 1_{[\tau_s \leq \tau_{s_0} \wedge n]} \wedge N_n^{(s)} \right)}{u+n\mu} \right)^{\alpha} \right].
\end{aligned}
\tag{2.56}
$$

Finally, the desired result follows from (2.47)- (2.49), (2.53), and (2.56).

3. A continuous time stationary SαS process associated with a conservative flow

In this section we consider a class of continuous-time claim processes $\mathbf{X}$ generated by a conservative flow. The construction of the class of such processes is due to [18]. In his paper, Samorodnitsky constructs a SαS random measure $M(\cdot)$ using a standard H-fractional Brownian motion, a centered, stationary increment Gaussian process, with self-similarity exponent $H \in (0,1)$. (See [19] or [7] for details on this process.) He then uses M to describe a SαS process $\mathbf{X}$ represented by a stochastic integral, and shows that this process is generated by a conservative flow for a certain class of kernels in the integral representation.

In this section we look at the Brownian motion case ($H = 1/2$), and we pick a fairly simple kernel in this class to show that even then (at least in the context of risk theory) the process is long-range dependent.

The continuous-time model in the insurance is of interest as an approximation in the presence of high-frequency claims which are irregularly spaced. The model can also be applied in the context of fluid queues and storage/dam processes. We continue to use the insurance risk theory language, however informally, and we give further details below:

3.1. Setup and preliminaries:

Let $\mathbf{B} = \{B(t), t \in \mathbb{R}\}$ be a standard Brownian motion (BM). Pick $E = C(-\infty, \infty)$, and let m be a σ-finite cylindrical measure on E defined by

$$m(A) = \int_{\mathbb{R}} P(\mathbf{B} \in A - y)\, dy, \quad A \text{ a cylindrical set,}$$

i.e. m is the (infinite) law of the BM shifted according to the Lebesgue measure on $\mathbb{R}$. Define

$$\varphi(x) := (1 - |x|) 1_{\{(1-|x|)\in[0,1]\}}, \quad x \in \mathbb{R}.$$

Note that $\varphi : \mathbb{R} \mapsto [0, \infty)$ is Hölder continuous with exponent 1, even, non-increasing on $[0, \infty)$, and $\varphi \in L^\alpha(\mathbb{R}, \mathcal{B}, \lambda)$. Clearly, the Hölder function

$$H(x) = \sup_{x \le s < t} \frac{\varphi(s) - \varphi(t)}{t - s}, \quad x \ge 0$$

also belongs to $L^\alpha(\mathbb{R}, \mathcal{B}, \lambda)$. Further define

$$X(t) = \int_E \varphi(x_t) M(d\mathbf{x}), \quad t \in \mathbb{R},\ \mathbf{x} = (x_s,\ s \in \mathbb{R}), \tag{3.1}$$

where M is a SαS random measure on E with control measure m. It is shown in [18] that the process $\mathbf{X} = \{X(t),\ t \in \mathbb{R}\}$ is a well defined stationary $S\alpha S$ process, and is generated by a conservative flow.

Now let the process $\mathbf{S} = \{S(t),\, t \ge 0\}$ be given by

$$S(t) := \int_0^t X(s)\, ds, \quad t \ge 0. \tag{3.2}$$

Notice that for any $T \in (0, \infty)$,

$$\begin{aligned} \int_0^T \left(\int_E \varphi(x_s)^\alpha m(d\mathbf{x}) \right)^{1/\alpha} ds &\le T \left(\sup_{0 \le s \le T} \int_E \varphi(x_s)^\alpha m(d\mathbf{x}) \right)^{1/\alpha} \\ &\le T \left(\int_E \sup_{0 \le s \le T} \varphi(x_s)^\alpha m(d\mathbf{x}) \right)^{1/\alpha}. \end{aligned} \tag{3.3}$$

But, it is shown in [18] that

$$b(T) := \left(\int_E \sup_{0 \le s \le T} \varphi(x_s)^\alpha m(d\mathbf{x}) \right)^{1/\alpha}$$

is finite. Thus it follows from Theorem 11.3.2 of [19] that

$$\int_0^T |X(s)|\, ds < \infty \ \text{ a.s.}$$

In particular, the process $\{S(t),\, t \in [0,T]\}$ is well-defined for any $T \in (0,\infty)$, and hence $\mathbf{S}$ is also well-defined.

Next let

$$h_t(\mathbf{x}) := \int_0^t \varphi(x_s)\, ds.$$

It follows from Theorem 11.4.1 of [19] that

$$S(t) = \int_E h_t(\mathbf{x})\, M(d\mathbf{x}) \ \text{ a.s.}, \quad t \ge 0. \tag{3.4}$$

Now, with $\mathbb{T} = \mathbb{R}_+$, the ruin probability given in (1.1) becomes

$$\psi(u) = P\left(\sup_{t \ge 0} (S(t) - \mu(t)) > u \right), \quad u > 0. \tag{3.5}$$

Lastly, for $u > 0$, let

$$\psi_0(u) := \frac{C_\alpha}{2} \int_{\mathbb{R}} \sup_{t \ge 0} \left(\frac{h_t(x)}{u + t\mu} \right)^\alpha dx = \frac{C_\alpha}{2} \int_{\mathbb{R}} E\left[\sup_{t \ge 0} \left(\frac{\int_0^t \varphi(B(s)+y)\, ds}{u + t\mu} \right)^\alpha \right] dy, \tag{3.6}$$

where $\mu > 0$ is the deterministic drift rate and $C_\alpha = (\int_0^\infty x^{-\alpha} \sin x\, dx)^{-1}$.

3.2. Asymptotic behavior of the ruin probability

We first prove the asymptotic equivalence of the ruin probability, $\psi(u)$, and $\psi_0(u)$ as u goes to infinity:

Proposition 3.1. *In the above setting*

$$\psi(u) \sim \psi_0(u) \quad \textit{as } u \to \infty. \tag{3.7}$$

Proof. We start with the following lemmas:

Lemma 3.1. *The following relation holds in the setting described above:*

$$\|h_t(\cdot)\|_{L^\alpha(E,\mathcal{E},m)} = O\left(t^{(\alpha+1)/2\alpha}\right), \quad t \to \infty.$$

Proof. Let $\{l(x,t),\, x \in \mathbb{R},\, t \geq 0\}$ be a jointly continuous local time process of $\mathbf{B}$ (see [7] for a brief definition or [3] for details.) As an immediate consequence of the self-similarity of the Brownian motion, local time process has the following scaling property: For any $c > 0$,

$$\left\{l\left(c^{1/2}x, ct\right),\, x \in \mathbb{R},\, t \geq 0\right\} \stackrel{d}{=} \left\{c^{1/2}l(x,t),\, x \in \mathbb{R},\, t \geq 0\right\}. \tag{3.8}$$

Moreover, all moments of $l(x,t)$ are finite, and are uniformly bounded in all real x and all real t in a compact set. (See for instance [5] for details.)

Now by Hölder's inequality and Fubini's theorem,

$$\begin{aligned}
\|h_t(\cdot)\|^\alpha_{L^\alpha(E,\mathcal{E},m)} &= \int_E h_t^\alpha(\mathbf{x})\, m(d\mathbf{x}) = \int_{\mathbb{R}} E\left[\left(\int_0^t \varphi(B(s)+y)\, ds\right)^\alpha\right] dy \\
&= \int_{\mathbb{R}} E\left[\left(\int_{\mathbb{R}} \varphi(x+y)\, l(x,t)\, dx\right)^\alpha\right] dy \leq \int_{\mathbb{R}} E\left[\left(\int_{-1-y}^{1-y} l(x,t)\, dx\right)^\alpha\right] dy \\
&\leq 2^{\alpha-1}\int_{\mathbb{R}} E\left(\int_{-1-y}^{1-y} l^\alpha(x,t)\, dx\right) dy = 2^\alpha \int_{\mathbb{R}} E\left[l^\alpha(x,t)\right] dx,
\end{aligned} \tag{3.9}$$

and by (3.8) we have

$$\begin{aligned}
\int_{\mathbb{R}} E\left[l^\alpha(x,t)\right] dx &= t^{\alpha/2}\int_{\mathbb{R}} E\left[l^\alpha\left(x/\sqrt{t},1\right)\right] dx \\
&= t^{\alpha/2}\int_{\mathbb{R}} E\left[l^\alpha\left(x/\sqrt{t},1\right) 1_{\left\{\sup\limits_{0\leq s\leq 1}|B(s)|\geq\left|x/\sqrt{t}\right|\right\}}\right] dx \\
&\leq t^{\alpha/2}\int_{\mathbb{R}} \left[E\left[l^2\left(x/\sqrt{t},1\right)\right]\right]^{\alpha/2}\left[P\left(\sup_{0\leq s\leq 1}|B(s)| \geq \left|\frac{x}{\sqrt{t}}\right|\right)\right]^{\frac{2-\alpha}{2}} dx \\
&\leq (constant)t^{(\alpha+1)/2}\int_{\mathbb{R}}\left[P\left(\sup_{0\leq s\leq 1}|B(s)| \geq |x|\right)\right]^{\frac{2-\alpha}{2}} dx.
\end{aligned} \tag{3.10}$$

(The last inequality is due to the fact that the moments of the local time are uniformly bounded.) Finally, the desired result follows by observing

$$\int_{\mathbb{R}} \left[P\left(\sup_{0\le s\le 1} |B(s)| \ge |x| \right) \right]^{\frac{2-\alpha}{2}} dx < \infty \tag{3.11}$$

as the supremum of a bounded Gaussian process has Gaussian-like tails. (See, for instance, [1].)

Lemma 3.2. *There exists $\tilde{\varepsilon} \in (0,1)$ such that the process $\tilde{\mathbf{Y}} = (\tilde{Y}(t),\ t \ge 0)$ defined by*

$$\tilde{Y}(t) := (t+1)^{\tilde{\varepsilon}-1} S(t), \quad t \ge 0,$$

is a.s. bounded.

Proof. It follows from Proposition 7.4 of [4] and Lemma 3.1 that there exists $\varepsilon_0 > 0$ such that the process

$$\big((n+1)^{\varepsilon_0 - 1} S(n),\ n = 0,1,2,\ldots\big)$$

is a.s. bounded.

Further, note by the stationarity of $\mathbf{X}$, for any $\tilde{\varepsilon} \in (0,1)$,

$$\begin{aligned} P&\left(\sup_{n=0,1,2,\ldots} \frac{\sup_{n\le t\le n+1} |S(t) - S(n)|}{(n+1)^{1-\tilde{\varepsilon}}} \ge \lambda \right) \\ &\le \sum_{n=0}^{\infty} P\left(\sup_{n\le t\le n+1} |S(t)-S(n)| \ge \lambda (n+1)^{1-\tilde{\varepsilon}} \right) \\ &= \sum_{n=0}^{\infty} P\left(\sup_{0\le t\le 1} |S(t)| \ge \lambda (n+1)^{1-\tilde{\varepsilon}} \right) \\ &\le \sum_{n=0}^{\infty} P\left(\sup_{0\le s\le 1} |X(s)| \ge \lambda (n+1)^{1-\tilde{\varepsilon}} \right) \end{aligned} \tag{3.12}$$

Also, it is shown in [18] that the process $\mathbf{X}$ is a.s. sample continuous. Consequently, $(X(s),\ s \in [0,1])$ is a.s. bounded. Then it follows from Theorem 10.5.1 of [19] that

$$\sum_{n=0}^{\infty} P\left(\sup_{0\le s\le 1} |X(s)| \ge \lambda (n+1)^{1-\tilde{\varepsilon}} \right) \le C \sum_{n=0}^{\infty} \left[\lambda (n+1)^{1-\tilde{\varepsilon}} \right]^{-\alpha},$$

for some positive constant C. Hence, we see that for any $\tilde{\varepsilon} < (1-\alpha^{-1})$,

$$\lim_{\lambda\to\infty} P\left(\sup_{n=0,1,2,\ldots} \frac{\sup_{n\le t\le n+1} |S(t)-S(n)|}{(n+1)^{1-\tilde{\varepsilon}}} \ge \lambda \right) = 0. \tag{3.13}$$

Consequently, for any such $\tilde{\varepsilon}$, it follows from monotone convergence theorem that the process

$$\left((n+1)^{\tilde{\varepsilon}-1} \sup_{n\le t\le n+1} |S(t)-S(n)|,\ n=0,1,2,\ldots \right)$$

is a.s. bounded.

Desired result follows by picking $\tilde{\varepsilon} \in (0, \min\{\varepsilon_0, (1-\alpha^{-1})\})$ and observing that

$$\sup_{t\ge 0} |\tilde{Y}(t)| \le \sup_{n=0,1,2,\ldots} (n+1)^{\tilde{\varepsilon}-1} |S(n)| + \sup_{n=0,1,2,\ldots} (n+1)^{\tilde{\varepsilon}-1} \sup_{n\le t\le n+1} |S(t)-S(n)|.$$

To finish the proof of the proposition pick $\tilde{\varepsilon} > 0$ such that $\tilde{\mathbf{Y}}$ is a.s. bounded and define a process $\mathbf{Y} = (Y(t),\ t\ge 0)$ by

$$Y(t) := \frac{[\log(t\mu+2)]^{1+\varepsilon}}{t\mu+2} S(t), \quad t\ge 0.$$

Note that for any $\varepsilon > 0$,

$$\frac{[\log(t\mu+2)]^{1+\varepsilon}}{t\mu+2} = o\left((t+1)^{\tilde{\varepsilon}-1}\right) \quad \text{as } t\to\infty.$$

Then, since $\tilde{\varepsilon} > 0$ is picked such that $\tilde{\mathbf{Y}}$ is a.s. bounded, we see for any $\varepsilon > 0$, $\mathbf{Y}$ is a.s. bounded. Now, the proposition follows from Theorem 4.1 and Remark 4.2 of [4].

What follows is the key step for the proof of the main theorem of this section:

Lemma 3.3. *For any $y \in \mathbb{R}$, as $u\to\infty$, the following relationship holds:*

$$g(u,y) := E\left[\sup_{t\ge 0} \left(\frac{\int_0^t \varphi(B(s)+y)ds}{u+t} \right)^{\alpha} \right] \sim u^{-\alpha/2} E\left[\sup_{t\ge 0} \left(\frac{l(0,t)}{1+t} \right)^{\alpha} \right]. \tag{3.14}$$

Proof. Fix $y \in \mathbb{R}$. For $K > 0$ start by defining

$$g^K(u,y) := E\left[\sup_{t \geq uK} \left(\frac{\int_0^t \varphi(B(s)+y)ds}{u+t}\right)^\alpha\right],$$

and

$$g_K(u,y) := E\left[\sup_{0 \leq t \leq uK} \left(\frac{\int_0^t \varphi(B(s)+y)ds}{u+t}\right)^\alpha\right].$$

Observe, by Hölder's inequality and Fubini's theorem,

$$\begin{aligned} g^K(u,y) &\leq \sum_{j=1}^\infty E\left[\sup_{uK2^{j-1} \leq t \leq uK2^j} \left(\frac{\int_0^t \varphi(B(s)+y)ds}{u+t}\right)^\alpha\right] \\ &\leq u^{-\alpha} \sum_{j=1}^\infty E\left(\frac{\int_0^{uK2^j} \varphi(B(s)+y)ds}{1+K2^{j-1}}\right)^\alpha \leq 2^\alpha u^{-\alpha} \sum_{j=1}^\infty E\left(\frac{\int_{-1-y}^{1-y} l(x,uK2^j)dx}{K2^j}\right)^\alpha, \end{aligned} \tag{3.15}$$

and by (3.8) and Hölder's inequality,

$$\begin{aligned} u^{-\alpha} \sum_{j=1}^\infty E\left(\frac{\int_{-1-y}^{1-y} l(x,uK2^j)dx}{K2^j}\right)^\alpha &= u^{-\alpha} \sum_{j=1}^\infty E\left[\frac{\sqrt{uK2^j}\int_{-1-y}^{1-y} l\left(\frac{x}{\sqrt{uK2^j}},1\right)dx}{K2^j}\right]^\alpha \\ &\leq 2^{\alpha-1} u^{-\alpha/2} K^{-\alpha/2} \sum_{j=1}^\infty 2^{-j\alpha/2} \int_{-1-y}^{1-y} E\left[l^\alpha\left(\frac{x}{\sqrt{uK2^j}},1\right)\right]dx. \end{aligned} \tag{3.16}$$

Then, it follows from the fact that the local time has moments of all orders finite and uniformly bounded in all real x,

$$\lim_{K\uparrow\infty} \limsup_{u\to\infty} u^{\alpha/2} g^K(u,y) = 0. \tag{3.17}$$

Next we will investigate $g_K(u,y)$. Start by noting that

$$\begin{aligned} \sup_{0 \leq t \leq uK} \frac{\sqrt{u}\int_0^t \varphi(B(s)+y)\,ds}{u+t} &\leq u^{-1/2} \int_0^{uK} \varphi(B(s)+y)\,ds \\ &\leq u^{-1/2} \int_0^{uK} 1_{\{B(s)\in[-1-y,1-y]\}}\,ds = u^{-1/2} \int_{-1-y}^{1-y} l(x,uK)\,dx, \end{aligned} \tag{3.18}$$

and it follows from Hölder's inequality that for any $\delta > 0$,

$$\left(\sup_{0 \le t \le uK} \frac{\sqrt{u} \int_0^t \varphi(B(s)+y)\, ds}{u+t} \right)^{\alpha+\delta} \le \frac{2^{\alpha+\delta-1}}{u^{(\alpha+\delta)/2}} \int_{-1-y}^{1-y} l^{\alpha+\delta}(x, uK)\, dx.$$

Consequently, by Fubini's theorem and (3.8) we have

$$\begin{aligned} \sup_{u>0} E \left(\left| \sup_{0 \le t \le uK} \frac{\sqrt{u} \int_0^t \varphi(B(s)+y)\, ds}{u+t} \right|^{\alpha+\delta} \right) \\ \le \sup_{u>0} \frac{2^{\alpha+\delta-1}}{u^{(\alpha+\delta)/2}} E \left(\int_{-1-y}^{1-y} l^{\alpha+\delta}(x, uK)\, dx \right) \\ = 2^{\alpha+\delta-1} \sup_{u>0} \int_{-1-y}^{1-y} E \left[l^{\alpha+\delta} \left(\frac{x}{\sqrt{u}}, K \right) \right] dx. \end{aligned} \tag{3.19}$$

But local time $l(x,t)$ has moments of all orders finite and uniformly bounded in all real x and all t in a compact set. Thus we conclude

$$\sup_{u>0} E \left(\left| \sup_{0 \le t \le uK} \frac{\sqrt{u} \int_0^t \varphi(B(s)+y)\, ds}{u+t} \right|^{\alpha+\delta} \right) < \infty,$$

and it follows from the "crystal ball condition" (c.f. p.184 of [15]) that for any $y \in \mathbb{R}$, the family

$$\left\{ \left(\sup_{0 \le t \le uK} \frac{\sqrt{u} \int_0^t \varphi(B(s)+y)\, ds}{u+t} \right)^{\alpha} \right\}_{u>0}$$

is uniformly integrable.

Next observe that

$$\left(u^{-1/2} \int_0^{ut} \varphi(B(s)+y)\, ds,\ t \ge 0 \right) \Rightarrow \left(l(0,t),\ t \ge 0 \right), \tag{3.20}$$

in $C[0,\infty)$ as $u \to \infty$. (See, for instance, p.52 of [7] for details.)

Thus, for any continuity point $z \ge 0$ of the distribution of $\sup_{0 \le v \le K} [l(0,v)/(1+v)]$, as $u \to \infty$,

$$\begin{aligned}
&P\left(\sup_{0\le t\le uK}\frac{u^H\int_0^t\varphi(B_H(s)+y)\,ds}{u+t}\ge z\right)\\
&\qquad= P\left(u^{H-1}\int_0^{uv}\varphi(B_H(s)+y)\,ds\ge(1+v)z\ \text{ for some } v\le K\right)\\
&\qquad\sim P\left(l(0,v)\ge(1+v)z\ \text{ for some } v\le K\right)\\
&\qquad= P\left(\sup_{0\le v\le K}\frac{l(0,v)}{1+v}\ge z\right).
\end{aligned}\tag{3.21}$$

Hence we conclude that as $u\to\infty$,

$$\sup_{0\le t\le uK}\frac{u^H\int_0^t\varphi(B_H(s)+y)\,ds}{u+t}\Rightarrow\sup_{0\le t\le K}\frac{l(0,t)}{1+t},\tag{3.22}$$

and therefore, by continuous mapping theorem,

$$\left(\sup_{0\le t\le uK}\frac{u^H\int_0^t\varphi(B_H(s)+y)\,ds}{u+t}\right)^\alpha\Rightarrow\left(\sup_{0\le t\le K}\frac{l(0,t)}{1+t}\right)^\alpha.\tag{3.23}$$

Now, recalling the uniform integrability, Theorem 6.6.1 of [15] implies,

$$\lim_{u\to\infty}u^{H\alpha}g_K(u,y)=E\left[\left(\sup_{0\le t\le K}\frac{l(0,t)}{1+t}\right)^\alpha\right],\tag{3.24}$$

and thus

$$\lim_{K\uparrow\infty}\lim_{u\to\infty}u^{H\alpha}g_K(u,y)=E\left[\left(\sup_{t\ge0}\frac{l(0,t)}{1+t}\right)^\alpha\right].\tag{3.25}$$

Lastly, recalling (3.17) we have

$$\lim_{u\to\infty}u^{H\alpha}g(u,y)=E\left[\left(\sup_{t\ge0}\frac{l(0,t)}{1+t}\right)^\alpha\right].\tag{3.26}$$

Now we state our theorem:

Theorem 3.1. *The following relation holds:*

$$\psi(u)\sim\frac{C_\alpha}{\sqrt{2\pi}}E\left[\sup_{t\ge0}\left(\frac{l(0,t)}{1+t}\right)^\alpha\right]\beta\left(\frac{1}{2},\frac{\alpha-1}{2}\right)\mu^{-\frac{1}{2}(\alpha+1)}u^{\frac{1}{2}(1-\alpha)},\quad u\to\infty,$$

where $\beta(\cdot,\cdot)$ is the Beta function.

Proof. By Proposition 3.1 it is sufficient to show the result for $\psi_0(u)$.

For $u > 0$ write

$$\begin{aligned} \frac{2\psi_0(u)}{C_\alpha} &= \int_{\mathbb{R}} E\left[\sup_{t>0}\left(\frac{\int_0^t \varphi(B(s)+y)\,ds}{u+t\mu}\right)^\alpha\right] dy \\ &= \int_{-\infty}^{-1} E\left[\sup_{t>0}\left(\frac{\int_0^t \varphi(B(s)+y)\,ds}{u+t\mu}\right)^\alpha\right] dy \\ &\qquad + \int_{-1}^{1} E\left[\sup_{t>0}\left(\frac{\int_0^t \varphi(B(s)+y)\,ds}{u+t\mu}\right)^\alpha\right] dy \\ &\qquad + \int_{1}^{\infty} E\left[\sup_{t>0}\left(\frac{\int_0^t \varphi(B(s)+y)\,ds}{u+t\mu}\right)^\alpha\right] dy \\ &=: I_1(u) + I_2(u) + I_3(u). \end{aligned} \tag{3.27}$$

Start by noting that by Hölder's inequality,

$$\begin{aligned} \limsup_{u\to\infty} u^{\frac{1}{2}(\alpha-1)} I_2(u) &= \limsup_{u\to\infty} u^{\frac{1}{2}(\alpha-1)} \int_{-1}^{1} E\left[\sup_{t>0}\left(\frac{\int_{\mathbb{R}} \varphi(x+y)\, l(x+t)\,dx}{u+t\mu}\right)^\alpha\right] dy \\ &\le 2 \limsup_{u\to\infty} u^{\frac{1}{2}(\alpha-1)} E\left[\sup_{t>0}\left(\frac{\int_{-2}^{2} l(x,t)\,dx}{u+t\mu}\right)^\alpha\right] \\ &\le 2^{2\alpha-1} \limsup_{u\to\infty} u^{\frac{1}{2}(\alpha-1)} E\left[\sup_{t>0}\frac{\int_{-2}^{2} l^\alpha(x,t)\,dx}{(u+t\mu)^\alpha}\right], \end{aligned} \tag{3.28}$$

and therefore by (3.8), and the fact that the supremum of the local time $l(x,t)$, for all real x and t in a compact set, has moments of all orders finite, we have

$$\begin{aligned} \limsup_{u\to\infty} u^{\frac{1}{2}(\alpha-1)} I_2(u) &\le 2^{2\alpha-1} \limsup_{u\to\infty} u^{\frac{1}{2}(\alpha-1)} E\left[\sup_{t>0}\frac{t^{\alpha/2}\int_{-2}^{2} l^\alpha(x/\sqrt{t},1)\,dx}{(u+t\mu)^\alpha}\right] \\ &\le (\textit{constant}) \limsup_{u\to\infty} u^{\frac{1}{2}(\alpha-1)} \sup_{t>0}\left(\frac{\sqrt{t}}{u+t\mu}\right)^\alpha \\ &= (\textit{constant}) \limsup_{u\to\infty} u^{\frac{1}{2}(\alpha-1)} \left(\frac{\sqrt{u/\mu}}{2u}\right)^\alpha = 0. \end{aligned} \tag{3.29}$$

Let $\tau[y] := \inf\{t \geq 0;\ B(t) = y\}$ be the *first passage time* to a level $y \in \mathbb{R}$, and observe

$$\begin{aligned} I_1(u) &= \int_{-\infty}^{-1} E\left[\sup_{t>\tau[-1-y]} \left(\frac{\int_0^t \varphi(B(s)+y)\,ds}{u+t\mu}\right)^\alpha\right] dy \\ &= \int_{-\infty}^{-1} E\left[\sup_{t>0} \left(\frac{\int_0^{t+\tau[-1-y]} \varphi(B(s)+y)\,ds}{u+(t+\tau[-1-y])\mu}\right)^\alpha\right] dy \\ &= \int_{-\infty}^{-1} E\left[\sup_{t>0} \left(\frac{\int_{\tau[-1-y]}^{t+\tau[-1-y]} \varphi(B(s)+y)\,ds}{u+(t+\tau[-1-y])\mu}\right)^\alpha\right] dy. \end{aligned} \tag{3.30}$$

Also recall that for $v > 0$ and $y \in \mathbb{R}$

$$P(\tau[y] \in dv) = \frac{|y|}{\sqrt{2\pi v^3}}\, e^{-y^2/2v} dv,$$

(c.f. p.80 [9].) Then it follows from the strong Markov property for Brownian motion and Fubini's theorem that

$$\begin{aligned} I_1(u) &= \int_{-\infty}^{-1}\int_0^\infty E\left[\sup_{t>0} \left(\frac{\int_0^t \varphi(B(s)-1)\,ds}{u+(t+v)\mu}\right)^\alpha\right] P(\tau[-1-y] \in dv)\,dy \\ &= \int_0^\infty E\left[\sup_{t>0} \left(\frac{\int_0^t \varphi(B(s)-1)\,ds}{u+(t+v)\mu}\right)^\alpha\right] \int_{-\infty}^{-1} \frac{-1-y}{\sqrt{2\pi v^3}}\, e^{-(-1-y)^2/2v} dy\,dv \\ &= \frac{1}{\mu^\alpha\sqrt{2\pi}} \int_0^\infty g(v+u/\mu,-1) v^{-1/2} dv \end{aligned} \tag{3.31}$$

Similarly,

$$I_3(u) = \frac{1}{\mu^\alpha\sqrt{2\pi}} \int_0^\infty g(v+u/\mu,1) v^{-1/2} dv \tag{3.32}$$

Now we will need the following Lemma:

Lemma 3.4. *For $y \in \mathbb{R}$, let*

$$I(u,y) := \int_0^\infty v^{-1/2} g(u+v,y)\,dv.$$

Then as $u \to \infty$

$$I(u,y) \sim u^{\frac{1}{2}(1-\alpha)} E\left[\sup_{t\ge 0}\left(\frac{l(0,t)}{1+t}\right)^{\alpha}\right] \beta\left(\frac{1}{2}, \frac{\alpha-1}{2}\right).$$

Proof. Pick $K > 0$. Define

$$I_1(u,y) := \int_{uK}^{\infty} v^{-1/2} g(u+v,y)\,dv, \text{ and } I_2(u,y) := \int_0^{uK} v^{-1/2} g(u+v,y)\,dv.$$

Note by monotonicity of g,

$$I_1(u,y) \le \int_{uK}^{\infty} v^{-1/2} g(v,y)\,dv.$$

Fix $\varepsilon > 0$. Then it follows from Lemma 3.3 that for sufficiently large u,

$$I_1(u,y) \le (1+\varepsilon) E\left[\sup_{t\ge 0}\left(\frac{l(0,t)}{1+t}\right)^{\alpha}\right] \int_{uK}^{\infty} v^{-(1+\alpha)/2}\,dv,$$

and hence

$$\lim_{K\uparrow\infty} \limsup_{u\to\infty} u^{\frac{1}{2}(\alpha-1)} I_1(u,y) = 0. \tag{3.33}$$

Also by Lemma 3.3 we have for any $K > 0$, as $u \to \infty$

$$\begin{aligned} I_2(u,y) &\sim E\left[\sup_{t\ge 0}\left(\frac{l(0,t)}{1+t}\right)^{\alpha}\right] \int_0^{uK} v^{-1/2}(u+v)^{-\alpha/2}\,dv \\ &= u^{\frac{1}{2}(1-\alpha)} E\left[\sup_{t\ge 0}\left(\frac{l(0,t)}{1+t}\right)^{\alpha}\right] \int_0^{K} x^{-1/2}(1+x)^{-\alpha/2}\,dx \end{aligned} \tag{3.34}$$

Desired result follows by letting $K \uparrow \infty$, taking (3.33) into account, and observing that

$$\int_0^{\infty} x^{-1/2}(1+x)^{-\alpha/2}\,dx = \beta\left(\frac{1}{2}, \frac{\alpha-1}{2}\right).$$

Finally note that

$$I_1(u) = \frac{1}{\mu^{\alpha}\sqrt{2\pi}}\, I(u/\mu, -1), \text{ and } I_3(u) = \frac{1}{\mu^{\alpha}\sqrt{2\pi}}\, I(u/\mu, 1),$$

and hence recalling (3.29) and using Lemma 3.4 we have

$$\begin{aligned} \frac{2\psi_0(u)}{C_\alpha} &= \frac{1}{\mu^\alpha \sqrt{2\pi}} \left[I(u/\mu, -1) + I(u/\mu, 1) \right] + o\left(u^{\frac{1}{2}(1-\alpha)} \right) \\ &\sim \frac{2}{\sqrt{2\pi}\mu^{\frac{1}{2}(\alpha+1)}} E\left[\sup_{t \geq 0} \left(\frac{l(0,t)}{1+t} \right)^\alpha \right] \beta\left(\frac{1}{2}, \frac{\alpha-1}{2} \right) u^{\frac{1}{2}(1-\alpha)}, \quad u \to \infty. \end{aligned} \tag{3.35}$$

Remark 3.1. All the results of this section prior to Theorem 3.1 are valid for general $H \in (0,1)$. This fact together with the observation of parallels between the main results of this section and the previous section lead us to believe that the result given in Theorem 3.1 should still hold with $1/2$ replaced by any $H \in (0,1)$. However our proof requires the use of strong Markov property which is only valid in the case where $H = 1/2$.

Acknowledgements

This research was supported in part by NSF training grant "Graduate and Postdoctoral Training in Probability and Its Applications," NSF grant DMS-0303493, and NSA grant MSPF-05G-049.

References

[1] Adler, R. J. (1990). *An introduction to continuity, extrema, and related topics for general Gaussian processes.* Institute of Mathematical Statistics Lecture Notes—Monograph Series, 12. Institute of Mathematical Statistics, Hayward, CA.

[2] Alparslan, U. T. and Samorodnitsky, G. (2006). Asymptotic analysis of exceedance probability with stationary stable steps generated by dissipative flows. In preparation.

[3] Berman, S. M. (1973/74). Local nondeterminism and local times of Gaussian processes. *Indiana Univ. Math. J.* **23,** 69–94.

[4] Braverman, M. (2004). Tail probabilities of subadditive functionals on stable processes with continuous and discrete time. *Stochastic Process. Appl.* **112,** 157–183.

[5] Cohen, S. and Samorodnitsky, G. (2006). Random rewards, fractional Brownian local times and stable self-similar processes. *Ann. Appl. Probab.* **16,** 1432–1461.

[6] Embrechts, P., Klüppelberg, C. and Mikosch, T. (1997). *Modelling extremal events* vol. 33 of *Applications of Mathematics (New York).* Springer-Verlag, Berlin. For insurance and finance.

[7] Embrechts, P. and Maejima, M. (2002). *Selfsimilar processes.* Princeton Series in Applied Mathematics. Princeton University Press, Princeton, NJ.

[8] Embrechts, P. and Veraverbeke, N. (1982). Estimates for the probability of ruin with special emphasis on the possibility of large claims. *Insurance Math. Econom.* **1,** 55–72.

[9] Karatzas, I. and Shreve, S. E. (1991). *Brownian motion and stochastic calculus* second ed. vol. 113 of *Graduate Texts in Mathematics.* Springer-Verlag, New York.

[10] Krengel, U. (1985). *Ergodic theorems* vol. 6 of *de Gruyter Studies in Mathematics.* Walter de Gruyter & Co., Berlin. With a supplement by Antoine Brunel.

[11] Lundberg, F. (1903). I approximerad framställning av sannolikhetsfunktionen. ii återförsäkring av kollektivrisker. *PhD thesis.* Akad. Afhandling. Almqvist och Wiksell, Uppsala.

[12] Mikosch, T. and Samorodnitsky, G. (2000). Ruin probability with claims modeled by a stationary ergodic stable process. *Ann. Probab.* **28,** 1814–1851.

[13] Resnick, S., Samorodnitsky, G. and Xue, F. (1999). How misleading can sample ACFs of stable MAs be? (Very!). *Ann. Appl. Probab.* **9,** 797–817.

[14] Resnick, S. I. (1992). *Adventures in stochastic processes.* Birkhäuser Boston Inc., Boston, MA.

[15] Resnick, S. I. (1999). *A probability path.* Birkhäuser Boston Inc., Boston, MA.

[16] Rosiński, J. (1995). On the structure of stationary stable processes. *Ann. Probab.* **23,** 1163–1187.

[17] Rosiński, J. and Samorodnitsky, G. (1996). Classes of mixing stable processes. *Bernoulli* **2,** 365–377.

[18] Samorodnitsky, G. (2004). Maxima of continuous-time stationary stable processes. *Adv. in Appl. Probab.* **36,** 805–823.

[19] Samorodnitsky, G. and Taqqu, M. S. (1994). *Stable non-Gaussian random processes.* Stochastic Modeling. Chapman & Hall, New York. Stochastic models with infinite variance.

[20] Teugels, J. L. (1970). Regular variation of Markov renewal functions. *J. London Math. Soc. (2)* **2,** 179–190.