The effect of memory on functional large deviations of infinite moving average processes

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Abstract

The large deviations of an infinite moving average process with exponentially light tails are very similar to those of an i.i.d. sequence as long as the coefficients decay fast enough. If they do not, the large deviations change dramatically. We study this phenomenon in the context of functional large, moderate and huge deviation principles.

Key words: large deviations, long range dependence, long memory, moving average, rate function, speed function

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1 Introduction

We consider (doubly) infinite moving average processes (X_n) defined by

$$X_n := \sum_{i=-\infty}^{\infty} \phi_i \, Z_{n-i}, n \in \mathbb{Z}, \tag{1.1}$$

where $\{Z_i, i \in \mathbb{Z}\}$ are i.i.d. $\mathbb{R}^d$-valued light-tailed random variables with 0 mean and covariance matrix Σ, and the coefficients (ϕ_i) are square summable:

$$\sum_{i=-\infty}^{\infty} \phi_i^2 < \infty. \tag{1.2}$$

Under these assumptions (X_n) is a well defined stationary process, also known as a linear process; see Brockwell and Davis (1991). Under a stronger assumption of absolute summability of coefficients,

$$\sum_{n \in \mathbb{Z}} |\phi_i| < \infty \tag{1.3}$$

the process is believed to have short memory; it is easy to check that the covariances are summable in this case: $\sum_{i=-\infty}^{\infty} |Cov(X_0, X_i)| < \infty$. It is also easy to exhibit a broad class of examples where (1.3) fails and the covariances are not summable.

Instead of covariances, we are interested in understanding how the large deviations of a moving average process change as the coefficients decay slower and slower. Information obtained this way is arguably more substantial than that obtained by covariances alone.

We assume that the moment generating function of a generic noise variable Z_0, is finite in a neighborhood of the origin, and then so is its log-moment generating function $\Lambda(\lambda) := \log E\big(\exp(\lambda \cdot Z_0)\big)$. Here $x \cdot y$ is the scalar product of two vectors x and y. Denoting for a function $f : \mathbb{R}^d \to (-\infty, \infty]$ the Fenchel-Legendre transform of f by f^*, and by $\mathcal{F}_f := \{x \in \mathbb{R}^d : f(x) < \infty\} \subset \mathbb{R}^d$, the assumption becomes $0 \in \mathcal{F}_\Lambda^\circ$, the interior of $\mathcal{F}_\Lambda$. Section 2.2 in Dembo and Zeitouni (1998) summarizes the properties of Λ and Λ^*.

We are interested in the large deviations of probability measures based on partial sums of a moving average process. Recall that a sequence of probability measures $\{\mu_n\}$ is said to satisfy the *large deviation principle*, or LDP, with speed b_n, and upper and lower rate function $I_u(\cdot)$ and $I_l(\cdot)$, respectively, if for any measurable set A,

$$-\inf_{x \in A^\circ} I_l(x) \le \liminf_{n\to\infty} \frac{1}{b_n} \log \mu_n(A) \le \limsup_{n\to\infty} \frac{1}{b_n} \log \mu_n(A) \le -\inf_{x \in \bar{A}} I_u(x); \tag{1.4}$$

a rate function is a non-negative lower semi-continuous function, and a good rate function is a rate function with compact level sets; we refer the reader to Varadhan (1984), Deuschel and Stroock (1989) or Dembo and Zeitouni (1998) for a detailed treatment of large deviations.

In the simplest case the sequence of measures $\{\mu_n\}$ is the sequence of the laws of the normalized sums $a_n^{-1}(X_1 + \ldots + X_n)$, where the normalization (a_n) is too large for a weak law to hold. Large deviations can also be formulated in function spaces, or in measure spaces. If the normaliztion satisfies $a_n = o(n)$, then one often refers to (1.4) as a moderate deviation principle, and the term "huge deviations" is often used in the case $n = o(a_n)$. This terminology reserves the term "proper large deviations" to the case where the sequence a_n grows linearly fast.

There exists rich literature on large deviation for the moving average processes, going back to Donsker and Varadhan (1985), who proved, for Gaussian moving averages, a LDP for he random measures $n^{-1}\sum_{i\le n} \delta_{X_i}$, under the assumption that the spectral density of the process is continuous at 0. Burton and Dehling (1990) considered a general moving average process and showed that, in the one-dimensional case with $\mathcal{F}_\Lambda = \mathbb{R}^1$, that under the assumption (1.3), if also

$$\sum_{n\in\mathbb{Z}} \phi_i = 1, \tag{1.5}$$

(or, more generally, the sum of the coefficients in not equal to zero), then $\{\mu_n\}$, the laws of the sample means, $n^{-1}S_n = n^{-1}(X_1 + \ldots + X_n)$, satisfy LDP with good rate function $\Lambda^*(\cdot)$. The work of Jiang et al. (1995) handled the case of $\{Z_i, i \in \mathbb{Z}\}$, taking values in a separable Banach space. Still assuming (1.3) and (1.5), they proved that the sequence $\{\mu_n\}$ satisfies a large deviation lower bound with the good rate function $\Lambda^*(\cdot)$, and under an integrability assumption that, in a finite dimensional Euclidian space, is equivalent to $0 \in \mathcal{F}_\Lambda^\circ$, a large deviation upper bound also holds, with a good rate function $\Lambda^{\#}(\cdot)$ given, once again in the finite dimensional case, by

$$\Lambda^{\#}(x) := \sup_{\lambda \in \Pi} \{\lambda \cdot x - \Lambda(\lambda)\}, \tag{1.6}$$

$$\text{where } \Pi = \{\lambda \in \mathbb{R}^d : \exists N_\lambda, \sup_{n \geq N_\lambda, i \in \mathbb{Z}} \Lambda(\lambda \phi_{i,n}) < \infty, \}$$

and $\phi_{i,n} := \phi_{i+1} + \cdots + \phi_{i+n}$. Observe that, if $\mathcal{F}_\Lambda = \mathbb{R}^d$ then $\Lambda^{\#} \equiv \Lambda^*$.

In their paper, Djellout and Guillin (2001) went back to the one-dimensional case, and worked under the assumption of a continuous at 0 and non-vanishing there spectral density. Assuming that the noise variables have a bounded support, they showed that the LDP of Burton and Dehling (1990) still holds, and also established a moderate deviation principle (with a Gaussian rate function).

Under the same assumption on the spectral density but in an arbitrary dimension $d \geq 1$, Wu (2004), replaced the assumption of the boundedness of the support of the noise variables by the strong integrability assumption $E[\exp(\delta |Z_0|^2)] < \infty$, for some δ in a neighborhood of 0. Under these assumptions an even stronger version of the large deviation principle, that for the occupation measures of the moving average processes, was shown to hold, with an explicit rate function under the absolute summability assumption (1.3).

Similarly, Jiang et al. (1992) considered moderate deviations in one dimension under the absolute summability of the coefficients, and assuming that $0 \in \mathcal{F}_\Lambda^\circ$. Finally, Dong et al. (2005) showed, under the same summability and integrability assumptions, that the moving average "inherits" its moderate deviations from the noise variables even if the latter are not necessarily i.i.d.

Oue main goal in this paper is to understand what happens when the absolute summability of the coefficients (or a variation, like existence of a continuous at the origin non-zero spectral density) fails. Specifically, we will assume a certain regular variation property of the coefficients; see Section 2. For contrast, we also present parallel results for the case where the coefficients are summable (most of the results are new even in this case). We will see that there is a significant difference between the large deviations in the case of absolutely summable coefficients, which are very similar to the large deviations of an i.i.d. sequence, and the situation we consider where this property fails. In this sense, there is justification behind taking (1.3), or "its neighbourhood", as the short memory range for moving average processes, and their complement as the long memory range. A similar phenomenon occurs in important applications to *ruin probabilities* and *long strange segments*; a discussion will appear in a companion paper.

The main part of the paper is Section 2, where we discuss functional LDP for a moving average process in both short and long memory settings. Certain lemmas required for the proofs in that section are postponed until Section 3.

2 Functional large deviation principle

This section discusses the sample path large, moderate and huge deviation principle for the moving average process. Specifically, we study the step process $\{Y_n\}$

$$Y_n(t) = \frac{1}{a_n}\sum_{i=1}^{[nt]} X_i, t \in [0,1], \tag{2.1}$$

and its polygonal path counterpart

$$\tilde{Y}_n(t) = \frac{1}{a_n}\sum_{i=1}^{[nt]} X_i + \frac{1}{a_n}(nt-[nt])X_{[nt]+1}, t \in [0,1]. \tag{2.2}$$

We will use the notation μ_n and $\tilde{\mu}_n$ to denote the laws of Y_n and $\tilde{Y}_n$ in the function space appropriate to the situation at hand, equipped with the cylindrical σ-field.

Various parts of the theorems will work with several topologies on the space $\mathcal{BV}$ of all $\mathbb{R}^d$-valued functions defined on the unit interval $[0,1]$ of bounded variation; to make sure the space $\mathcal{BV}$ is a measurable set in the cyindrical σ-field, we use only rational partitions of $[0,1]$ when defining variation. We will use subscripts to denote the topology on the space. Specifically, the subscripts S, P and L will denote the sup-norm topology, the topology of pointwise convergence and, finally, the topology in which f_n converges to f if and only if f_n converges to f both pointwise and in L_p for all $p \in [1,\infty)$.

We call a function $f : \mathbb{R}^d \to \mathbb{R}$ *balanced regular varying* with exponent $\beta > 0$, if there exists a non-negative bounded function ζ_f defined on the unit sphere on $\mathbb{R}^d$ and a function $\tau_f : [0,\infty) \to [0,\infty)$ satisfying

$$\lim_{t\to\infty} \frac{\tau_f(tx)}{\tau_f(t)} = x^\beta \tag{2.3}$$

for all $x > 0$ (i.e. τ_f is regularly varying with exponent β) and such that for any $(\lambda_t) \subset \mathbb{R}^d$ with $|\lambda_t| = 1$ for all t, converging to λ,

$$\lim_{t\to\infty} \frac{f(t\lambda_t)}{\tau_f(t)} = \zeta_f(\lambda). \tag{2.4}$$

We will typically omit the subscript f if doing so is not likely to cause confusion.

The following assumption describes the short memory scenarios we consider. In addition to summability of the coefficients, the different cases arise from the "size" of the normalizing constants (a_n) in (2.1), the resulting speed sequence (b_n) and the integrability assumptions on the noise variables.

Assumption 2.1. *All the scenarios below assume that*

$$\sum_{i\in\mathbb{Z}} |\phi_i| < \infty \text{ and } \sum_{i\in\mathbb{Z}} \phi_i = 1. \tag{2.5}$$

*S*1. $a_n = n, 0 \in \mathcal{F}_\Lambda^\circ$ *and* $b_n = n$.

*S*2. $a_n = n, \mathcal{F}_\Lambda = \mathbb{R}^d$ *and* $b_n = n$.

*S*3. $a_n/\sqrt{n} \to \infty,\ a_n/n \to 0,\ 0 \in \mathcal{F}_\Lambda^\circ$ *and* $b_n = a_n^2/n$.

*S*4. $a_n/n \to \infty$, $\Lambda(\cdot)$ *is balanced regular varying with exponent* $\beta > 1$ *and* $b_n = n\tau(\gamma_n)$, *where*

$$\gamma_n = \sup\{x : \tau(x)/x \le a_n/n\}. \tag{2.6}$$

Let $\phi_{i,n} := \phi_{i+1} + \cdots + \phi_{i+n}$ for all $i \in \mathbb{Z}$ and $n \ge 1$. Also for any $k \ge 1$ and $0 < t_1 < \cdots < t_k \le 1$, let $\Pi_{t_1,\ldots,t_k} \subset (\mathbb{R}^d)^k$ be defined as

$$\Pi_{t_1,\ldots,t_k} := \Big\{\underline{\lambda} = (\lambda_1,\ldots,\lambda_k) \in (\mathcal{F}_\Lambda)^k :\ \Lambda \text{ is continuous on } \mathcal{F}_\Lambda \text{ at } \lambda_j,\ j = 1,\ldots,k$$

$$\text{and for some } N = 1,2,\ldots \sup_{n\ge N,\, j\in\mathbb{Z}} \Lambda\Big(\sum_{i=1}^k \lambda_i \phi_{j+[nt_i],[nt_i]-[nt_{i-1}]}\Big) < \infty\Big\}. \tag{2.7}$$

We view the next theorem as describing the sample path large deviations of (the partial sums of) a moving average process in the short memory case. The long memory counterpart is Theorem 2.4 below.

Theorem 2.2. *(i) If S*1 *holds, then* $\{\mu_n\}$ *satisfy in* $\mathcal{BV}_L$, *LDP with speed* $b_n \equiv n$, *good upper rate function*

$$G^{sl}(f) = \sup_{k\ge1,\, t_1,\ldots,t_k} \Big\{ \sup_{\underline{\lambda}\in\Pi_{t_1,\ldots,t_k}} \sum_{i=1}^k \Big\{\lambda_i\cdot\big(f(t_i)-f(t_{i-1})\big)-(t_i-t_{i-1})\Lambda(\lambda_i)\Big\}\Big\} \tag{2.8}$$

if $f(0) = 0$ *and* $G^{sl}(f) = \infty$ *otherwise, and with good lower rate function*

$$H^{sl}(f) = \begin{cases} \int_0^1 \Lambda^*(f'(t))dt & if \quad f \in \mathcal{AC}, f(0) = 0 \\ \infty & otherwise, \end{cases}$$

where $\mathcal{AC}$ *is the set of all absolutely continuous functions, and* f' *is the coordinate-wise derivative of* f.

*(ii) If S*2 *holds, then* $H^{sl} \equiv G^{sl}$ *and* $\{\mu_n\}$ *satisfy LDP in* $\mathcal{BV}_S$, *with speed* $b_n \equiv n$ *and good rate function* $H^{sl}(\cdot)$.

(iii) *Under assumption S3, $\{\mu_n\}$ satisfy in $\mathcal{BV}_S$, LDP with speed b_n and good rate function*

$$H^{sm}(f) = \begin{cases} \int_0^1 \frac{1}{2} f'(t) \cdot \Sigma^{-1} f'(t) dt & if \quad f \in \mathcal{AC}, f(0) = 0 \\ \infty & otherwise, \end{cases}$$

Here Σ is the covariance matrix of Z_0, and we understand $a \cdot \Sigma^{-1} a$ to mean ∞ if $a \in K_\Sigma := \{\lambda \in \mathbb{R}^d - \{0\} : \Sigma\lambda = 0\}$.

(iv) *Under assumption S4, $\{\mu_n\}$ satisfy in $\mathcal{BV}_S$, LDP with speed b_n and good rate function*

$$H^{sh}(f) = \begin{cases} \int_0^1 (\Lambda^h)^*(f'(t)) dt & if \quad f \in \mathcal{AC}, f(0) = 0 \\ \infty & otherwise, \end{cases} .$$

where $\Lambda^h(\lambda) = \zeta_\Lambda\left(\frac{\lambda}{|\lambda|}\right)|\lambda|^\beta$ for $\lambda \in \mathbb{R}^d$ (defined as zero for $\lambda = 0$).

A comparison with the LDP for i.i.d. sequences (see Mogulskii (1976) or Theorem 5.1.2 in Dembo and Zeitouni (1998)) reveals that the rate function stays the same as long as the coefficients in the moving average process stay summable.

We also note that an application of the contraction principle gives, under scenario S1, a marginal LDP for the law of $n^{-1}S_n$ in $\mathbb{R}^d$ with speed n, upper rate function $G_1^{sl}(x) = \sup_{\lambda\in\Pi_1}\left\{\lambda \cdot x - \Lambda(\lambda)\right\}$, and lower rate function $\Lambda^*(\cdot)$, recovering the statement of Theorem 1 in Jiang et al. (1995) in the finite-dimensional case.

Next, we consider what happens when the absolute summability fails, in a "major way". We will assume that the coefficients are balanced regular varying with an appropriate exponent. The following assumption is parallel to Assumption 2.1 in the present case, dealing, once again, with the various cases that may arise.

Assumption 2.3. *All the scenarios assume that the coefficients $\{\phi_i\}$ are balanced regular varying with exponent $-\alpha, 1/2 < \alpha \leq 1$ and $\sum_{i=-\infty}^{\infty} |\phi_i| = \infty$. Specifically, there is $\psi : [0, \infty) \to [0, \infty)$ and $0 \leq p \leq 1$, such that*

$$\left.\begin{array}{c} \lim_{t\to\infty} \frac{\psi(tx)}{\psi(t)} = x^{-\alpha}, \ for \ all \ x > 0 \\ \lim_{n\to\infty} \frac{\phi_n}{\psi(n)} = p \ and \ \lim_{n\to\infty} \frac{\phi_{-n}}{\psi(n)} = q := 1 - p. \end{array}\right\} \tag{2.9}$$

Let $\Psi_n := \sum_{1\leq i\leq n} \psi(i)$.

R1. $a_n = n\Psi_n, 0 \in \mathcal{F}_\Lambda^\circ$ and $b_n = n$.

R2. $a_n = n\Psi_n, \mathcal{F}_\Lambda = \mathbb{R}^d$ and $b_n = n$.

R3. $a_n/\sqrt{n}\Psi_n \to \infty, a_n/(n\Psi_n) \to 0, 0 \in \mathcal{F}_\Lambda^\circ$ *and* $b_n = a_n^2/(n\Psi_n^2)$.

R4. $a_n/(n\Psi_n) \to \infty$, $\Lambda(\cdot)$ *is balanced regular varying with exponent* $\beta > 1$ *and* $b_n = n\tau(\Psi_n\gamma_n)$, *where*

$$\gamma_n = \sup\{x : \tau(\Psi_n x)/x \le a_n/n\}. \tag{2.10}$$

Similar to (2.7) we define

$$\Pi^\alpha_{t_1,\dots,t_k} := \Big\{ \underline{\lambda} = (\lambda_1,\dots,\lambda_k) : (p \wedge q)\lambda_i \in \mathcal{F}_\Lambda^\circ,\ i = 1,\dots,k, \text{ and}$$

$$\text{for some } N = 1,2,\dots \sup_{n \ge N,\, j\in\mathbb{Z}} \Lambda\Big(\frac{1}{\Psi_n} \sum_{i=1}^k \lambda_i \phi_{j+[nt_i],[nt_i]-[nt_{i-1}]} \Big) < \infty \Big\} \tag{2.11}$$

for $1/2 < \alpha < 1$, while for $\alpha = 1$, we define

$$\Pi^1_{t_1,\dots,t_k} := \Big\{ \underline{\lambda} = (\lambda_1,\dots,\lambda_k) \in (\mathcal{F}_\Lambda)^k :\ \Lambda \text{ is continuous on } \mathcal{F}_\Lambda \text{ at each } \lambda_j$$

$$\text{and for some } N = 1,2,\dots \sup_{n \ge N,\, j\in\mathbb{Z}} \Lambda\Big(\frac{1}{\Psi_n} \sum_{i=1}^k \lambda_i \phi_{j+[nt_i],[nt_i]-[nt_{i-1}]} \Big) < \infty \Big\} \tag{2.12}$$

Also for $1/2 < \alpha < 1$, any $k \ge 1$, $0 < t_1 \le \cdots \le t_k \le 1$, and $\underline{\lambda} = (\lambda_1,\dots,\lambda_k) \in (\mathbb{R}^d)^k$ let

$$h_{t_1,\dots,t_k}(x;\underline{\lambda}) := (1-\alpha) \sum_{i=1}^k \lambda_i \int_{x+t_{i-1}}^{x+t_i} |y|^{-\alpha} (pI_{[y\ge 0]} + qI_{[y<0]})dy. \tag{2.13}$$

For any $\mathbb{R}^d$-valued convex function Γ, any function $\varphi \in L_1[0,1]$ and $1/2 < \alpha < 1$ we define ,

$$\Gamma^*_\alpha(\varphi) = \sup_{\psi \in L_\infty[0,1]} \Big\{ \int_0^1 \psi(t)\cdot\varphi(t)\,dt \tag{2.14}$$

$$- \int_{-\infty}^{\infty} \Gamma\left(\int_0^1 \psi(t)(1-\alpha)|x+t|^{-\alpha} \Big[pI_{[x+t\ge 0]} + qI_{[x+t<0]} \Big] dt \right) dx \Big\},$$

whereas for $\alpha = 1$ we put

$$\Gamma^*_1(\varphi) = \int_0^1 \Gamma^*(\varphi(t))\,dt\,. \tag{2.15}$$

We view the following result as describing the large deviations of moving averages in the long memory case.

Theorem 2.4. *(i) If $R1$ holds, then $\{\mu_n\}$ satisfy in $\mathcal{BV}_L$, LDP with speed $b_n = n$, good upper rate function*

$$G^{rl}(f) = \sup_{k\geq 1, t_1,\dots,t_k} \Big\{ \sup_{\underline{\lambda}\in \Pi^{\alpha}_{t_1,\dots,t_k}} \sum_{i=1}^{k} \lambda_i \cdot \big(f(t_i) - f(t_{i-1})\big) - \Lambda^{rl}_{t_1,\dots,t_k}(\lambda_1,\dots,\lambda_k) \Big\} \tag{2.16}$$

if $f(0) = 0$ and $G^{rl}(f) = \infty$ otherwise, where

$$\Lambda^{rl}_{t_1,\cdots,t_k}(\lambda_1,\cdots,\lambda_k) \quad := \quad \begin{cases} \int_{-\infty}^{\infty} \Lambda\Big(h_{t_1,\dots,t_k}(x;\underline{\lambda})\Big)dx & if \quad \alpha < 1 \\ \sum_{i=1}^{k} (t_i - t_{i-1})\Lambda(\lambda_i) & if \quad \alpha = 1, \end{cases} \tag{2.17}$$

and good lower rate function

$$H^{rl}(f) = \begin{cases} \Lambda^*_\alpha(f') & if \quad f \in \mathcal{AC}, f(0) = 0 \\ \infty & otherwise. \end{cases}$$

(ii) If $R2$ holds, then $H^{rl} \equiv G^{rl}$ and $\{\mu_n\}$ satisfy LDP in $\mathcal{BV}_S$, with speed $b_n = n$ and good rate function $H^{rl}(\cdot)$.

(iii) Under assumption $R3$, $\{\mu_n\}$ satisfy in $\mathcal{BV}_S$, LDP with speed b_n and good rate function

$$H^{rm}(f) = \begin{cases} (G_\Sigma)^*_\alpha(f') & if \quad f \in \mathcal{AC}, f(0) = 0 \\ \infty & otherwise, \end{cases}$$

where $G_\Sigma(\lambda) = \frac{1}{2}\lambda\cdot\Sigma\lambda$, $\lambda \in \mathbb{R}^d$.

(iv) Under assumption $R4$, $\{\mu_n\}$ satisfy in $\mathcal{BV}_S$, LDP with speed b_n and good rate function

$$H^{rh}(f) = \begin{cases} (\Lambda^h)^*_\alpha(f') & if \quad f \in \mathcal{AC}, f(0) = 0 \\ \infty & otherwise, \end{cases}$$

with Λ^h as in Theorem 2.2.

We note that a functional LDP for a non-stationary fractional ARIMA model (corresponding to our assumption $R2$) was obtained by Barbe and Broniatowski (1998).

Remark 2.5. The proof of Theorem 2.4 below shows that, under the assumption $R1$, the laws of $(n\Psi_n)^{-1}S_n$ satisfy LDP with speed n, good lower rate function $\Lambda^{rl*}_1(\cdot)$ and good upper rate function $G^{rl}_1(x) := \sup_{\lambda\in\Pi^\alpha_1} \big\{\lambda\cdot x - \Lambda^{rl}_1(\lambda)\big\}$. If $R2$ holds, then $\Pi^\alpha_1 = \mathbb{R}^d$ and $G^{rl}_1 \equiv (\Lambda^{rl}_1)^*$.

Remark 2.6. It is interesting to note that under the assumption $R3$ it is possible to choose $a_n = n$, and compare the large deviations of the sample means of moving average processes with summable and non-summable coefficients. We see that the former, under assumptions S1 or S2, satisfy LDP with speed $b_n = n$, while the latter satisfy LDP with speed $b_n = n/\Psi_n^2$, which is regular varying with exponent $2\alpha - 1$. The markedly slower speed function in the latter case (even for $\alpha = 1$ one has $b_n = nL_n$, with a slowly varying function L converging to zero) demonstrates a phase transition occurring here.

Remark 2.7. Lemma 2.8 at the end of this section describes certain properties of the rate function $(G_\Sigma)^*_\alpha$, which is, clearly, also the rate function in *all* scenarios in the Gaussian case.

The proofs of theorems 2.2 and 2.4 rely on lemmas appearing in Section 3.

Proof of Theorem 2.2. (ii), (iii) and (iv) Let $\mathcal{X}$ be the set of all $\mathbb{R}^d$-valued functions defined on the unit interval $[0,1]$ and let $\mathcal{X}^o$ be the subset of $\mathcal{X}$, of functions which start at the origin. Define J as the collection of all ordered finite subsets of $(0,1]$ with a partial order defined by inclusion. For any $j = \{0 < t_1 < \ldots < t_{|j|} \leq 1\}$ define the projection $p_j : \mathcal{X}^o \to \mathcal{Y}_j$ as $p_j(f) = (f(t_1), \ldots, f(t_{|j|}))$, for any $f \in \mathcal{X}^o$. So $\mathcal{Y}_j$ can be identified with the space $(\mathbb{R}^d)^{|j|}$ and the projective limit of $\mathcal{Y}_j$ over $j \in J$ can be identified with $\mathcal{X}^o$ equipped with the topology of pointwise convergence. Note that $\mu_n \circ p_j^{-1}$ is the law of

$$Y_n^j = (Y_n(t_1), \ldots, Y_n(t_{|j|}))$$

and let

$$V_n = \big(Y_n(t_1), Y_n(t_2) - Y_n(t_1), \cdots, Y_n(t_{|j|}) - Y_n(t_{|j|-1})\big). \tag{2.18}$$

By Lemma 3.5 we get that for any $\underline{\lambda} = (\lambda_1, \ldots, \lambda_{|j|}) \in (\mathbb{R}^d)^{|j|}$

$$\lim_{n\to\infty} \frac{1}{b_n} \log E\big(\exp\big[b_n \underline{\lambda}\cdot V_n\big]\big) = \lim_{n\to\infty} \frac{1}{b_n} \log E \exp\Big[\frac{b_n}{a_n} \sum_{i=1}^{|j|} \lambda_i \cdot \Big(\sum_{k=[nt_{i-1}]+1}^{[nt_i]} X_k\Big)\Big]$$

$$= \lim_{n\to\infty} \frac{1}{b_n} \sum_{l=-\infty}^{\infty} \Lambda\Big(\frac{b_n}{a_n} \sum_{i=1}^{|j|} \lambda_i \phi_{l+[nt_{i-1}],[nt_i]-[nt_{i-1}]}\Big)$$

$$= \sum_{i=1}^{|j|} (t_i - t_{i-1}) \Lambda^v(\lambda_i) := \Lambda^v_{t_1,\ldots,t_{|j|}}(\underline{\lambda}),$$

where $t_0 = 0$ and for any $\lambda \in \mathbb{R}^d$,

$$\Lambda^v(\lambda) = \begin{cases} \Lambda(\lambda) & \text{in part (ii)}, \\ \frac{1}{2}\lambda \cdot \Sigma\lambda & \text{in part (iii)}, \\ \zeta\Big(\frac{\lambda}{|\lambda|}\Big)|\lambda|^\beta & \text{in part (iv)}. \end{cases}$$

By the Gartner-Ellis theorem, the laws of (V_n) satisfy LDP with speed b_n and good rate function

$$\Lambda^{v*}_{t_1,\cdots,t_{|j|}}(w_1,\ldots,w_{|j|}) = \sum_{i=1}^{|j|}(t_i - t_{i-1})\Lambda^{v*}\Big(\frac{w_i}{t_i - t_{i-1}}\Big)$$

where $(w_1,\ldots,w_{|j|}) \in (\mathbb{R}^d)^{|j|}$. The map $V_n \mapsto Y_n^j$ from $(\mathbb{R}^d)^{|j|}$ onto itself is one to one and continuous. Hence the contraction principle tells us that $\{\mu_n \circ p_j^{-1}\}$ satisfy LDP in $(\mathbb{R}^d)^{|j|}$ with good rate function

$$H^v_{t_1,\ldots,t_{|j|}}(y_1,\ldots,y_{|j|}) := \sum_{i=1}^{|j|}(t_i - t_{i-1})\Lambda^{v*}\Big(\frac{y_i - y_{i-1}}{t_i - t_{i-1}}\Big), \tag{2.19}$$

where we take $y_0 = 0$. By Lemma 3.1, the same holds for the measures $\{\tilde{\mu}_n \circ p_j^{-1}\}$. Proceeding as in Lemma 5.1.6 in Dembo and Zeitouni (1998) this implies that the measures $\{\tilde{\mu}_n\}$ satisfy LDP in the space $\mathcal{X}^o$ equipped with the topology of pointwise convergence, with speed b_n and the rate function described in the appropriate part of the theorem. As $\mathcal{X}^o$ is a closed subset of $\mathcal{X}$, the same holds for $\{\tilde{\mu}_n\}$ in $\mathcal{X}$ and the rate function is infinite outside $\mathcal{X}^o$. Since $\tilde{\mu}_n(\mathcal{BV}) = 1$ for all $n \geq 1$ and the 3 rate functions in parts (ii), (iii) and (iv) of the theorem are infinite outside of $\mathcal{BV}$, we conclude that $\{\tilde{\mu}_n\}$ satisfy LDP in $\mathcal{BV}_P$ with the same rate function. The sup-norm topology on $\mathcal{BV}$ in stronger than that of pointwise convergence and by Lemma 3.2, $\{\tilde{\mu}_n\}$ is exponentially tight in $\mathcal{BV}_S$. So by Corollary 4.2.6 in Dembo and Zeitouni (1998), $\{\tilde{\mu}_n\}$ satisfy LDP in $\mathcal{BV}_S$ with speed b_n and good rate function $H^v(\cdot)$. Finally, applying Lemma 3.1 once again, we conclude that the same is true for the sequence $\{\mu_n\}$.

(i): We use the above notation. It follows from Lemma 3.5 that for any partition j of $(0,1]$ and $\underline{\lambda} = (\lambda_1,\ldots,\lambda_{|j|}) \in (\mathbb{R}^d)^{|j|}$,

$$\limsup_{n\to\infty} \frac{1}{n} \log E\big[\exp\big(n\underline{\lambda}\cdot V_n\big)\big] \leq \chi(\underline{\lambda}),$$

where

$$\chi(\underline{\lambda}) = \begin{cases} \sum_{i=1}^{|j|}(t_i - t_{i-1})\Lambda(\lambda_i) & \text{if } \underline{\lambda} \in \Pi_{t_1,\ldots,t_{|j|}} \\ \infty & \text{otherwise.} \end{cases}$$

The law of V_n is exponentially tight since by Jiang et al. (1995) the law of $Y_n(t_i) - Y_n(t_{i-1})$ is exponentially tight in $\mathbb{R}^d$ for every $1 \leq i \leq |j|$. Thus by Theorem 2.1 of de Acosta (1985) the laws of (V_n) satisfy the LD upper bound with speed n and rate function

$$\sup_{\underline{\lambda}\in\Pi_{t_1,\ldots,t_{|j|}}} \left\{ \underline{\lambda}\cdot\underline{w} - \sum_{i=1}^{|j|}(t_i - t_{i-1})\Lambda(\lambda_i) \right\},$$

which is, clearly, good. Therefore, the laws of $(Y_n(t_1),\dots,Y_n(t_{|j|}))$ satisfy the LD upper bound with speed n and good rate function

$$G^{sl}_{t_1,\dots,t_{|j|}}(\underline{y}) := \sup_{\underline{\lambda}\in\Pi_{t_1,\dots,t_{|j|}}} \Big\{\sum_{i=1}^{|j|}\lambda_i\cdot(y_i-y_{i-1})-\sum_{i=1}^{|j|}(t_i-t_{i-1})\Lambda(\lambda_i)\Big\}. \tag{2.20}$$

Using the upper bound part of the Dawson-Gartner theorem, we see that $\{\mu_n\}$ satisfy LD upper bound in $\mathcal{X}_P^o$ with speed n and a good rate rate function

$$G^{sl}(f)=\sup_{j\in J} G^{sl}_{t_1,\dots,t_{|j|}}\big(f(t_1),\cdots,f(t_{|j|})\big),$$

and as before, the same holds in $\mathcal{X}_P$, as well.

Next we prove that $(Y_n(t_1),\dots,Y_n(t_{|j|}))$ satisfy LD lower bound with speed n and rate function $H^v_{t_1,\dots,t_{|j|}}(\cdot)$ defined in (2.19) for part (ii). Let

$$V_n' = \frac{1}{n}\Big(\sum_{|i|\le 2n}\phi_{i,[nt_1]}Z_{-i}, \sum_{|i|\le 2n}\phi_{i+[nt_1],[nt_2]-[nt_1]}Z_{-i},\cdots,$$
$$\sum_{|i|\le 2n}\phi_{i+[nt_{|j|-1}],[nt_{|j|}]-[nt_{|j|-1}]}Z_{-i}\Big)$$

and observe that the laws of (V_n) and of (V_n') are exponentially equivalent.

For $k>0$ large enough so that $p_k := P(|Z_0|\le k)>0$ we let $\mu_k = E\big(Z_0\big|\,|Z_0|\le k\big)$, and note that $|\mu_k|\to 0$ as $k\to\infty$. Let

$$V_n'^{,k} = \frac{1}{n}\Big(\sum_{|i|\le 2n}\phi_{i,[nt_1]}(Z_{-i}-\mu_k), \sum_{|i|\le 2n}\phi_{i+[nt_1],[nt_2]-[nt_1]}(Z_{-i}-\mu_k),\cdots,$$
$$\sum_{|i|\le 2n}\phi_{i+[nt_{|j|-1}],[nt_{|j|}]-[nt_{|j|-1}]}(Z_{-i}-\mu_k))\Big) := V_n' - a_{n,k}\,,$$

where $a_{n,k}=(b_1^{(n)}\mu_k, b_2^{(n)}\mu_k,\dots,b_{|j|}^{(n)}\mu_k)\in(R^d)^{|j|}$ with some $|b_i^{(n)}|\le c$, a constant independent of i and n. We define a new probability measure

$$\nu_n^k(\cdot) = P\Big(V_n'^{,k}\in\cdot, |Z_i|\le k,\ \text{for all } |i|\le 2n\Big)p_k^{-(4n+1)}\,.$$

Note that for all $\underline{\lambda}\in(\mathbb{R}^d)^{|j|}$ by (the proof of part (i) of) Lemma 3.5,

$$\lim_{n\to\infty}\tfrac{1}{n}\log\Big\{p_k^{-(4n+1)}E\Big[\exp\big(n\underline{\lambda}\cdot V_n'\big)I_{[|Z_i|\le k,\ |i|\le 2n]}\Big]\Big\}$$
$$=\sum_{l=1}^{|j|}(t_l-t_{l-1})\Big(L^k(\lambda_l)-\lambda_l\mu_k\Big)-t_{|j|}\log p_k,$$

where $L^k(\lambda) := \log E\big[\exp(\lambda\cdot Z_0)I_{[|Z_0|\le k]}\big]$, and so for every $k\ge 1$, $\{\nu_n^k, n\ge 1\}$ satisfy LDP with speed n and good rate function

$$\sup_{\underline{\lambda}}\Big\{\underline{\lambda}\cdot\underline{x}-\sum_{l=1}^{|j|}(t_l-t_{l-1})\Big(L^k(\lambda_l)-\lambda_l\mu_k\Big)\Big\}+t_{|j|}\log p_k$$
$$=\sum_{l=1}^{|j|}(t_l-t_{l-1})L^{k*}\Big(\tfrac{x_l+t_{|j|}\mu_k}{t_l-t_{l-1}}\Big)+t_{|j|}\log p_k\,. \tag{2.21}$$

Since for any open set G

$$\liminf_{n\to\infty}\frac{1}{n}\log P(V_n^{\prime,k}\in G)\geq\liminf_{n\to\infty}\frac{1}{n}\log\nu_n^k(G)+4\log p_k\,,$$

we conclude that for any x and $\epsilon>0$, for all k large enough,

$$\liminf_{n\to\infty}\frac{1}{n}\log P(V_n'\in B(\underline{x},2\epsilon))\geq\liminf_{n\to\infty}\frac{1}{n}\log\nu_n^k(B(\underline{x},\epsilon))+4\log p_k\,,$$

where $B(\underline{x},\epsilon)$ is an open ball centered at x with radius ϵ.

Now note that for every $\lambda\in\mathbb{R}^d$, $L^k(\lambda)$ is increasing to $\Lambda(\lambda)$ with k. So by theorem B3 in de Acosta (1988), there exists $\{\underline{x^k}\}\subset(\mathbb{R}^d)^{|j|}$, such that $\underline{x^k}\to\underline{x}$, and

$$\limsup_{k\to\infty}\sum_{l=1}^{|j|}(t_l-t_{l-1})L^{k*}\Big(\frac{x_l^k}{t_l-t_{l-1}}\Big)\leq\sum_{l=1}^{|j|}(t_l-t_{l-1})L^{*}\Big(\frac{x_l}{t_l-t_{l-1}}\Big)\,.$$

Since $\underline{x^k}-t_{|j|}\underline{\mu}_k\in B(\underline{x},2\epsilon)$ for k large, where $\underline{\mu}_k=(\mu_k,\ldots,\mu_k)\in(R^d)^{|j|}$, we conclude that

$$\liminf_{n\to\infty}\frac{1}{n}\log P(V_n'\in B(\underline{x},\epsilon))\geq-\sum_{l=1}^{|j|}(t_l-t_{l-1})\Lambda^{*}\Big(\frac{x_l}{t_l-t_{l-1}}\Big)\,.$$

Furthermore, because the laws of (V_n) and of (V_n') are exponentially equivalent, the same statement holds with V_n replacing V_n'. We have, therefore, established that the laws of $(Y_n(t_1),\ldots,Y_n(t_{|j|}))$ satisfy LD lower bound with speed n and good rate function $H^v_{t_1,\ldots,t_{|j|}}(\cdot)$ defined in (2.19) for part (ii). By the lower bound part of the Dawson-Gärtner theorem, $\{\mu_n\}$ satisfy a LD lower bound in $\mathcal{X}_P$ with speed n and rate function $\sup_{j\in J}H^v_{t_1,\ldots,t_{|j|}}(f(t_1),\,\ldots,f(t_{|j|}))$. This rate function is identical to H^{sl}.

Notice that the lower rate function H^{sl} is infinite outside of the space $\cap_{p\in[1,\infty)}L_p[0,1]$, and by Lemma 3.4, the same is true for the upper rate function G^{sl} (we view $\cap_{p\in[1,\infty)}L_p[0,1]$ as a measurable subset of $\mathcal{X}$ with respect to the universal completion of the cylindrical σ-field). We conclude that the measures $\{\mu_n\}$ satisfy a LD lower bound in $\cap_{p\in[1,\infty)}L_p[0,1]$ with the topology of pointwise convergence. Since this topology is coarser than the L topology, we can use Lemma 3.3 to conclude that the LD upper bound and the LD lower bound also hold in $\cap_{p\in[1,\infty)}L_p[0,1]$ equipped with L topology. Finally, the rate functions are also infnite outside of the space $\mathcal{BV}$, and so the measures $\{\mu_n\}$ satisfy the LD bounds in $\mathcal{BV}$ equipped with L topology. □

Proof of Theorem 2.4. The proof of parts (ii), (iii) and (iv) is identical to the proof of the corresponding parts in Theorem 2.2, except that Lemma 3.6 is used now instead of Lemma 3.5, and we use Lemma 3.8 to identify the rate function.

We now prove part (i) of the theorem. We start by proving the finite dimensional LDP for the laws of V_n in (2.18). An inspection of the proof of the corresponding statement on Theorem 2.2 shows that the only missing ingredient needed to obtain the upper bound part of this LDP is the exponential tightness of $Y_n(1)$ in $\mathbb{R}^d$. Notice that for $s > 0$ and small $\lambda > 0$

$$P\Big(Y_n(1) \notin [-s,s]^d\Big) \le e^{-\lambda n s} \sum_{l=1}^{d} E\Big(e^{\lambda Y_n^{(l)}(1)} + e^{-\lambda Y_n^{(l)}(1)}\Big),$$

where $Y_n^{(l)}(1)$ is the lth coordinate of $Y_n(1)$. Since $0 \in \mathcal{F}_\Lambda^o$, by part (i) of Lemma 3.6 we see that

$$\lim_{s\to\infty} \limsup_{n\to\infty} \frac{1}{n} \log P\Big(Y_n(1) \notin [-s,s]^d\Big) = -\infty\,,$$

which is the required exponential tightness. It follows that the laws of (V_n) satisfy the LD upper bound with speed n and rate function

$$\sup_{\underline{\lambda} \in \Pi^{rl}_{t_1,\ldots,t_{|j|}}} \Big\{\underline{\lambda} \cdot \underline{w} - \Lambda^{rl}_{t_1,\ldots,t_{|j|}}(\lambda_1,\ldots,\lambda_{|j|})\Big\}\,.$$

Next we prove the LD lower bound for the laws of (V_n). The proof in the case $\alpha = 1$ follows the same steps as the corresponding argument in Theorem 2.2, so we will concentrate on the case $1/2 < \alpha < 1$. For $m \ge 1$ let

$$V'_{n,m} = \frac{1}{n\Psi_n}\Big(\sum_{|i|\le mn} \phi_{i,[nt_1]} Z_{-i}, \sum_{|i|\le mn} \phi_{i+[nt_1],[nt_2]-[nt_1]} Z_{-i}, \cdots,$$

$$\sum_{|i|\le mn} \phi_{i+[nt_{|j|-1}],[nt_{|j|}]-[nt_{|j|-1}]} Z_{-i}\Big)\,.$$

Observe that $V_n = V'_{n,m} + R'_{n,m}$ for some $R'_{n,m}$ independent of $V'_{n,m}$ and such that for every m, $R'_{n,m} \to 0$ in probability as $n \to \infty$. We conclude that for any $\underline{x} = (x_1, \cdots, x_{|j|}) \in (\mathbb{R}^d)^{|j|}$, $\epsilon > 0$, and n sufficiently large, one has

$$P(V_n \in B(\underline{x}, 2\epsilon)) \ge \frac{1}{2} P(V'_{n,m} \in B(\underline{x},\epsilon))\,. \tag{2.22}$$

For $k \ge 1$ we define p_k and μ_k as in the proof of Theorem 2.2, and once again we choose k large enough so that $p_k > 0$. We also define

$$V'^{,k}_{n,m} = \frac{1}{n\Psi_n}\Big(\sum_{|i|\le mn} \phi_{i,[nt_1]}\big(Z_{-i} - \mu_k\big), \sum_{|i|\le mn} \phi_{i+[nt_1],[nt_2]-[nt_1]}\big(Z_{-i} - \mu_k\big), \cdots,$$

$$\sum_{|i|\le mn} \phi_{i+[nt_{|j|-1}],[nt_{|j|}]-[nt_{|j|-1}]}\big(Z_{-i} - \mu_k\big)\Big) := V'_{n,m} - a^{(m)}_{n,k}\,,$$

where $a^{(m)}_{n,k} = (b_1^{(n,m)}\mu_k, b_2^{(n,m)}\mu_k, \ldots, b_{|j|}^{(n,m)}\mu_k) \in (R^d)^{|j|}$ with some $|b_i^{(n,m)}| \le c_m$, a constant independent of i and n.

Once again we define a new probability measure by

$$\nu_n^{k,m}(\cdot) = P\Big(V'^{,k}_{n,m} \in \cdot, |Z_i| \le k, \text{ for all } |i| \le mn\Big) p_k^{-(2mn+1)} .$$

Note that for all $\underline{\lambda} \in (\mathbb{R}^d)^{|j|}$, by (the proof of) Lemma 3.6,

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{n} \log \Big\{ p_k^{-(2mn+1)} E\Big[\exp\big(n\underline{\lambda} \cdot V'^{,k}_{n,m}\big) I_{[|Z_i| \le_k,\, |i| \le mn]} \Big] \Big\} \\
= & \int_{-m}^{m} L^k \left((1-\alpha) \sum_{i=1}^{|j|} \lambda_i \int_{x+t_{i-1}}^{x+t_i} |y|^{-\alpha} (pI_{[y\ge 0]} + qI_{[y<0]}) dy \right) dx \\
- & (1-\alpha) \sum_{l=1}^{|j|} \lambda_i \cdot \mu_k \int_{-m}^{m} \left(\int_{x+t_{i-1}}^{x+t_i} |y|^{-\alpha} (pI_{[y\ge 0]} + qI_{[y<0]}) dy \right) dx - 2m \log p_k \\
= & Q^{k,m}(\underline{\lambda}) - \mu_k \cdot R^m(\underline{\lambda}) - 2m \log p_k \text{ (say)}
\end{aligned}$$

where $L^k(\lambda) = \log E\big[\exp(\lambda \cdot Z_0) I_{[|Z_0| \le k]}\big]$, as defined before. Therefore, for every $k \ge 1$, $\{\nu_n^{k,m}, n \ge 1\}$ satisfy LDP with speed n and good rate function $(Q^{k,m})^*(\underline{x} - \underline{c}_{k,m}) + 2m \log p_k$, where $\underline{c}_{k,m} = (c_1^m \mu_k, c_2^m \mu_k, \dots, c_{|j|}^m \mu_k) \in (\mathbb{R}^d)^{|j|}$ with

$$c_i^m = (1-\alpha) \int_{-m}^{m} \left(\int_{x+t_{i-1}}^{x+t_i} |y|^{-\alpha} (pI_{[y\ge 0]} + qI_{[y<0]}) dy \right) dx .$$

Note that for every $\lambda \in \mathbb{R}^d$, $L^k(\lambda)$ is increasing to $\Lambda(\lambda)$ and $Q^{k,m}(\underline{\lambda})$ is increasing to

$$\Lambda^{rl,m}_{t_1,\dots,t_{|j|}}(\underline{\lambda}) = \int_{-m}^{m} \Lambda\Big(h_{t_1,\dots,t_k}(x;\underline{\lambda})\Big) dx$$

with k.

An application of Theorem B3 in de Acosta (1988) shows, as in the proof of Theorem 2.2, that for any ball centered at x with radius ϵ

$$\liminf_{n\to\infty} \frac{1}{n} \log P(V'_{n,m} \in B(\underline{x}, \epsilon)) \ge -(\Lambda^{rl,m}_{t_1,\dots,t_{|j|}})^*(\underline{x}).$$

Appealing to (2.22) gives us

$$\liminf_{n\to\infty} \frac{1}{n} \log P(V_n \in B(\underline{x}, 2\epsilon)) \ge -(\Lambda^{rl,m}_{t_1,\dots,t_{|j|}})^*(\underline{x})$$

for all $m \ge 1$. We now apply the above argument once again: for every $\lambda \in \mathbb{R}^d$, $\Lambda^{rl,m}_{t_1,\dots,t_{|j|}}(\underline{\lambda})$ increases to $\Lambda^{rl}_{t_1,\dots,t_{|j|}}(\underline{\lambda})$, and yet another appeal to Theorem B3 in de Acosta (1988) gives us the desired LD lower bound for the laws of (V_n) in the case $1/2 < \alpha < 1$.

Continuing as in the proof of Theorem 2.2 we conclude that $\{\mu_n\}$ satisfy a LD lower bound in $\mathcal{X}_P$ with speed n and rate function $\sup_{j\in J}(\Lambda^{rl}_{t_1,\dots,t_{|j|}})^*(f(t_1),$

$f(t_2) - f(t_1), \ldots, f(t_{|j|}) - f(t_{|j|-1})$). By Lemma 3.8 this is equal to $H^{rl}(f)$ in the case $1/2 < \alpha < 1$, and in the case $\alpha = 1$ the corresponding statement is the same as in Theorem 2.2. The fact that the LD lower bound holds also in $\mathcal{BV}_L$ follows in the same way as in Theorem 2.2. This completes the proof. $\square$

The next lemma discusses some properties of the rate function $(G_\Sigma)^*_\alpha$ in Theorem 2.4. For $0 < \theta < 1$, let

$$H_\theta = \left\{ \psi : [0,1] \to \mathbb{R}^d, \text{ measurable, and } \int_0^1 \int_0^1 \frac{|\psi(t)||\psi(s)|}{|t-s|^\theta} dt\, ds < \infty \right\}.$$

If Σ is a nonnegative definite matrix, we define an inner product on H_θ by

$$(\psi_1, \psi_2)_\Sigma = \int_0^1 \int_0^1 \frac{\psi_1(t) \cdot \Sigma \psi_2(s)}{|t-s|^\theta} dt\, ds.$$

This results in an incomplete inner product space; see Landkof (1972). Observe also that $L_\infty[0,1] \subset H_\theta \subset L_2[0,1]$, and that

$$(\psi_1, \psi_2)_\Sigma = (\psi_1, T_\theta \psi_2),$$

where

$$(\psi_1, \psi_2) = \int_0^1 \psi_1(t) \cdot \psi_2(t)\, dt$$

is the inner product in $L_2[0,1]$, and $T_\theta : H_\theta \to H_\theta$ is defined by

$$T_\theta \psi(t) = \int_0^1 \frac{\Sigma \psi(s)}{|t-s|^\theta} ds. \tag{2.23}$$

Lemma 2.8. *For $\varphi \in L_1[0,1]$ and $1/2 < \alpha < 1$,*

$$(G_\Sigma)^*_\alpha(\varphi) = \sup_{\psi \in L_\infty[0,1]} (\psi, \varphi) - \frac{\sigma^2}{2}\Big(\psi, T_{2\alpha-1}\psi\Big), \tag{2.24}$$

where

$$\sigma^2 = (1-\alpha)^2 \int_{-\infty}^{\infty} |x+1|^{-\alpha}|x|^{-\alpha} \Big[pI_{[x+1\ge 0]} + qI_{[x+1<0]}\Big]\Big[pI_{[x\ge 0]} + qI_{[x<0]}\Big]\, dx,$$

ψ is regarded as an element of the dual space $L_1[0,1]'$, and $T_{2\alpha-1}$ in (2.23) is regarded as a map $L_\infty[0,1] \to L_1[0,1]$.

(i) Suppose that $\varphi \in T_{2\alpha-1}H_{2\alpha-1}$. Then

$$(G_\Sigma)^*_\alpha(\varphi) = \frac{1}{2\sigma^2}\|h\|^2_\Sigma,$$

where $\varphi = T_{2\alpha-1}h$.

(ii) Suppose that $Leb\{t \in [0,1] : \varphi(t) \in K_\Sigma\} > 0$, where $K_\Sigma = Ker(\Sigma) - \{0\}$ is as defined in (2.2). Then $(G_\Sigma)^*_\alpha(\varphi) = \infty$.

Proof. Note that for $\varphi \in L_1[0,1]$

$$\int_{-\infty}^{\infty} G_\Sigma\left(\int_0^1 \psi(t)(1-\alpha)|x+t|^{-\alpha}\Big[pI_{[x+t\geq 0]} + qI_{[x+t<0]}\Big]\,dt\right)$$

$$= \frac{1}{2}(1-\alpha)^2 \int_0^1 \int_0^1 \psi(s)\cdot\Sigma\psi(t)\bigg(\int_{-\infty}^{\infty} |x+s|^{-\alpha}|x+t|^{-\alpha}\Big[pI_{[x+s\geq 0]} + qI_{[x+s<0]}\Big]$$

$$\Big[pI_{[x+t\geq 0]} + qI_{[x+t<0]}\Big]\,dx\bigg)ds\,dt = \frac{\sigma^2}{2}\int_0^1\int_0^1 \frac{\psi(s)\cdot\Sigma\psi(t)}{|t-s|^\theta}ds\,dt\,,$$

and so (2.24) follows.

For part (i), suppose that $\varphi = T_{2\alpha-1}h$ for $h \in H_{2\alpha-1}$. For $\psi \in H_{2\alpha-1}$ we have

$$(\psi,\varphi) - \frac{\sigma^2}{2}(\psi, T_{2\alpha-1}\psi) = \frac{1}{2\sigma^2}\big(h, T_{2\alpha-1}h\big) - \frac{\sigma^2}{2}\Big((\psi - \frac{1}{\sigma^2}h), T_{2\alpha-1}(\psi - \frac{1}{\sigma^2}h)\Big)$$

because the operator $T_{2\alpha-1}$ is self-adjoint. Therefore,

$$\sup_{\psi\in H_{2\alpha-1}} (\psi,\varphi) - \frac{\sigma^2}{2}(\psi, T_{2\alpha-1}\psi) = \frac{1}{2\sigma^2}\big(h, T_{2\alpha-1}h\big)\,,$$

achieved at $\psi_0 = h/\sigma^2$, and so by (2.24),

$$(G_\Sigma)^*_\alpha(\varphi) \leq \frac{1}{2\sigma^2}\big(h, T_{2\alpha-1}h\big)\,.$$

On the other hand, for $M > 0$ let $\psi_0^{(M)} = \psi_0\mathbf{1}\big(|\psi_0| \leq M\big) \in L_\infty[0,1]$. Then

$$(G_\Sigma)^*_\alpha(\varphi) \geq \limsup_{M\to\infty} \psi_0^{(M)}(\varphi) - \frac{\sigma^2}{2}\psi_0^{(M)}\Big(T_{2\alpha-1}\psi_0^{(M)}\Big)$$

$$= (\psi_0,\varphi) - \frac{\sigma^2}{2}(\psi_0, T_{2\alpha-1}\psi_0) = \frac{1}{2\sigma^2}\big(h, T_{2\alpha-1}h\big)\,,$$

completing the proof of part (i).

For part (ii), note that using (2.24) and choosing for $c > 0$ $\psi(t) = c\varphi(t)/|\varphi(t)|$ if $\varphi(t) \in K_\Sigma$, and $\psi(t) = 0$ otherwise, we obtain

$$(G_\Sigma)^*_\alpha(\varphi) \geq c\int_A |\varphi(t)|\,dt\,,$$

where $A = \{t \in [0,1] : \varphi(t) \in K_\Sigma\}$. The proof is completed by letting $c \to \infty$. □

3 Lemmas and their proofs

In this section we prove the lemmas used in Section 2, the notation of which is retained here (often without mentioning it explicitely).

Lemma 3.1. *Under any of the assumptions S2, S3, S4, R2, R3 or R4, the families* $\{\mu_n\}$ *and* $\{\tilde{\mu}_n\}$ *are exponentially equivalent in* $\mathcal{D}_S$*, where* $\mathcal{D}$ *is the space of all right-continuous functions with left limits and, as before, the subscript denotes the topology on that space.*

Proof. It is clearly enough to consider the case $d = 1$. For any $\delta > 0$ and $\lambda \in \mathcal{F}_\Lambda \cap -\mathcal{F}_\Lambda$, $\lambda \neq 0$,

$$\begin{aligned}
\limsup_{n\to\infty} \frac{1}{b_n} \log P\Big(||Y_n - \tilde{Y}_n|| > \delta\Big) &\leq \limsup_{n\to\infty} \frac{1}{b_n} \log P\Big(\frac{1}{a_n} \max_{1\leq i\leq n} |X_i| > \delta\Big) \\
&\leq \limsup_{n\to\infty} \frac{1}{b_n} \log \Big(n\, P(|X_1| > a_n\delta)\Big) \\
&\leq \limsup_{n\to\infty} \frac{1}{b_n} \Big(\log n - a_n\lambda\delta + \Lambda(\lambda) + \Lambda(-\lambda))\Big) \\
&= \limsup_{n\to\infty} \frac{1}{b_n} \Big(- a_n\lambda\delta\Big) .
\end{aligned}$$

Under the assumptions S3, S4, R3 or R4 we have $a_n/b_n \to \infty$, so the above limit is equal to $-\infty$. Under the assumptions S2 and R2, $a_n = b_n$, but we can let $\lambda \to \infty$ after taking the limit in n. □

Lemma 3.2. *Under any of the assumptions S2, S3, S4, R2, R3 or R4, the family* $\{\tilde{\mu}_n\}$ *is exponentially tight in* $\mathcal{D}_S$*, i.e, for every* $\pi > 0$ *there exists a compact* $K_\pi \subset \mathcal{D}_S$*, such that*

$$\lim_{\pi\to\infty} \limsup_{n\to\infty} \frac{1}{b_n} \log \tilde{\mu}_n(K_\pi^c) = -\infty.$$

Proof. We first prove the lemma assuming that $d = 1$. We use the notation $w(f, \delta) := \sup_{s,t\in[0,1],|s-t|<\delta} |f(s) - f(t)|$ for the modulus of continuity of a function $f : [0, 1] \to \mathbb{R}^d$. First we claim that for any $\epsilon > 0$,

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \frac{1}{b_n} \log P\Big(w(\tilde{Y}_n, \delta) > \epsilon\Big) = -\infty, \tag{3.1}$$

where $\tilde{Y}_n$ is the polygonal process in (2.2). Let us prove the lemma assuming that the claim is true. By (3.1) and the continuity of the paths of $\tilde{Y}_n$, there is $\delta_k > 0$ such that for all $n \geq 1$

$$P\Big(w(\tilde{Y}_n, \delta_k) \geq k^{-1}\Big) \leq e^{-\pi b_n k},$$

and set $A_k = \{f \in \mathcal{D} : w(f, \delta_k) < k^{-1}, f(0) = 0\}$. Now the set $K_\pi := \overline{\cap_{k\geq 1} A_k}$ is compact in $\mathcal{D}_S$ and by union of events bound it follows that

$$\limsup_{n\to\infty} \frac{1}{b_n} \log P(\tilde{Y}_n \notin K_\pi) \leq -\pi,$$

establishing the exponential tightness. Next we prove the claim (3.1). Observe that for any $\epsilon > 0$, $\delta > 0$ small and $n > 2/\delta$

$$\begin{aligned}
P & \Big(w(\tilde{Y}_n, \delta) > \epsilon\Big) \leq P\Big(\max_{0\leq i<j\leq n, j-i\leq [n\delta]+2} \frac{1}{a_n}\Big|\sum_{k=i}^{j} X_k\Big| > \epsilon\Big) \\
\leq & \; n\sum_{i=1}^{[2n\delta]} P\Big(\frac{1}{c_n}\Big|\sum_{k=1}^{i} X_k\Big| > b_n\epsilon\Big) \\
\leq & \; ne^{-b_n\lambda\epsilon}\sum_{i=1}^{[2n\delta]} E\Big[\exp\Big(\frac{\lambda}{c_n}\sum_{k=1}^{i} X_k\Big) + \exp\Big(-\frac{\lambda}{c_n}\sum_{k=1}^{i} X_k\Big)\Big] \\
= & \; ne^{-b_n\lambda\epsilon}\sum_{i=1}^{[2n\delta]} \Big(\exp\Big[\sum_{j\in\mathbb{Z}} \Lambda\Big(\frac{\lambda}{c_n}\phi_{j,i}\Big)\Big] + \exp\Big[\sum_{j\in\mathbb{Z}} \Lambda\Big(-\frac{\lambda}{c_n}\phi_{j,i}\Big)\Big]\Big) \\
\leq & \; \frac{2n^2\delta}{e^{b_n\lambda\epsilon}}\Big(\exp\Big[\sum_{j\in\mathbb{Z}} \Lambda\Big(\frac{|\lambda|}{c_n}|\phi|_{j,[2n\delta]}\Big)\Big] + \exp\Big[\sum_{j\in\mathbb{Z}} \Lambda\Big(-\frac{|\lambda|}{c_n}|\phi|_{j,[2n\delta]}\Big)\Big]\Big)
\end{aligned}$$

by convexity of Λ, where $|\phi|_{i,n} = |\phi_{i+1}| + \cdots + |\phi_{i+n}|$ for $i \in \mathbb{Z}$ and $n \geq 1$.

Therefore by lemmas 3.5 and 3.6 we get

$$\lim_{\delta\to 0}\limsup_{n\to\infty} \frac{1}{b_n}\log P\big(w(\tilde{Y}_n, \delta) > \epsilon\big) \leq -\lambda\epsilon.$$

Now, letting $\lambda \to \infty$ we obtain (3.1).

If $d \geq 1$ then $\{\tilde{\mu}_n\}$ is exponentially tight since $\{\tilde{\mu}_n^k\}$, the law of the kth coordinate of $\tilde{Y}_n$, is exponentially tight for every $1 \leq k \leq d$. □

Lemma 3.3. *Under the assumptions $S1$ or $R1$ the family $\{\mu_n\}$ is, for any $p \in [1, \infty)$, exponentially tight in the space of functions in $\cap_{p\in[1,\infty)} L_p[0,1]$, equipped with the topology L, where f_n converges to f if and only if f_n converges to f both pointwise and in $L_p[0,1]$ for all $p \in [1, \infty)$.*

Proof. Here $a_n = n$ under the assumption $S1$, $a_n = n\Psi_n$ under the assumption $R1$, and $b_n = n$ in both cases. As before, it is enough to consider the case $d = 1$. We claim that for any $p \in [1, \infty)$,

$$\begin{aligned}
\lim_{x\downarrow 0}\limsup_{n\to\infty} \frac{1}{n}\log P\Big[\int_0^{1-x} \Big|Y_n(t+x) - Y_n(t)\Big|^p dt \qquad (3.2) \\
+ \int_0^x \Big|Y_n(t)\Big|^p dt + \int_{1-x}^1 \Big|Y_n(t)\Big|^p dt > \epsilon\Big] = -\infty,
\end{aligned}$$

for any $\epsilon > 0$, while

$$\lim_{M\uparrow\infty}\limsup_{n\to\infty} \frac{1}{n}\log P\Big(\sup_{0\leq t\leq 1} |Y_n(t)| > M\Big) = -\infty\,. \qquad (3.3)$$

Assuming that both claims are true, for any $\pi > 0$, $m \geq 1$ and $k \geq 1$, we can choose (using the fact that $Y_n \in L^\infty[0,1]$ a.s. for all $n \geq 1$) $0 < x_k^{(m)} < 1$ such that for all $n \geq 1$,

$$P\Big[\int_0^{1-x_k^{(m)}} \Big|Y_n(t+x_k^{(m)}) - Y_n(t)\Big|^m dt$$

$$+\int_0^{x_k^{(m)}} \Big|Y_n(t)\Big|^m dt + \int_{1-x_k^{(m)}}^1 \Big|Y_n(t)\Big|^m dt > k^{-1}\Big] \leq e^{-\pi knm},$$

and $M_\pi > 0$ such that for all $n \geq 1$

$$P\Big(\sup_{0\leq t\leq 1} |Y_n(t)| > M_\pi\Big) \leq e^{-\pi n}.$$

Now define sets

$$A_{k,m} = \Big\{f \in \cap_{p\geq 1} L_p[0,1] : \int_0^{1-x_k^{(m)}} \Big|f(t+x_k^{(m)}) - f(t)\Big|^m dt$$

$$+\int_0^{x_k^{(m)}} \Big|f(t)\Big|^m dt + \int_{1-x_k^{(m)}}^1 \Big|f(t)\Big|^m dt \leq k^{-1},\ \sup_{0\leq t\leq 1} |f(t)| \leq M_\pi\Big\},$$

and set $K_\pi = \overline{\cap_{k,m\geq 1} A_{k,m}}$. Then K_π is compact for every $\pi > 0$ by Tychonov's theorem (see Theorem 19, p. 166 in Royden (1968)) and Theorem 20, p. 298 in Dunford and Schwartz (1988). Furthermore,

$$\limsup_{n\to\infty} \frac{1}{b_n} \log P[Y_n \notin K_\pi] \leq -\pi.$$

This will complete the proof once we prove (3.2) and (3.3). We first prove (3.2) for $p = 1$. Observe that

$$P\Big[\int_0^{1-x} |Y_n(t+x) - Y_n(t)| dt > \epsilon\Big] \leq P\Big[\frac{[nx]}{n}\frac{1}{a_n}\sum_{i=1}^n |X_i| > \epsilon\Big]$$

$$\leq e^{-\lambda n\epsilon/x} E\Big[\exp\Big(\lambda\frac{1}{c_n}\sum_{i=1}^n |X_i|\Big)\Big] \leq e^{-\lambda n\epsilon/x} E\Big[\prod_{i=1}^n \exp\Big(\frac{\lambda}{c_n}|X_i|\Big)\Big]$$

$$\leq e^{-\lambda n\epsilon/x} E\Big[\prod_{i=1}^n \Big(\exp\Big(\frac{\lambda}{c_n}X_i\Big) + \exp\Big(-\frac{\lambda}{c_n}X_i\Big)\Big)\Big]$$

$$= e^{-\lambda n\epsilon/x} \sum_{l_i=\pm 1} E\Big[\exp\Big(\frac{\lambda}{c_n}\sum_{i=1}^n l_i X_i\Big)\Big]$$

$$= e^{-\lambda n\epsilon/x} \sum_{l_1=\pm 1} \exp\Big(\sum_{j\in\mathbb{Z}} \Lambda\Big(\frac{\lambda}{c_n}(\phi_{j+1}l_1 + \cdots + \phi_{j+n}l_n)\Big)\Big)$$

$$\leq e^{-\lambda n\epsilon/x} 2^n \exp\Big(\sum_{j\in\mathbb{Z}} \Lambda\Big(\frac{\lambda}{c_n}|\phi|_{j,n}\Big) + \sum_{j\in\mathbb{Z}} \Lambda\Big(-\frac{\lambda}{c_n}|\phi|_{j,n}\Big)\Big).$$

Therefore,

$$\begin{aligned} &\limsup_{n\to\infty} \frac{1}{n} \log P\Big[\int_0^{1-x} |Y_n(t+x_k) - Y_n(t)|dt > \epsilon\Big] \\ \leq \quad & -\lambda\frac{\epsilon}{x} + \log 2 + \limsup_{n\to\infty} \frac{1}{n}\sum_{j\in\mathbb{Z}} \Lambda\Big(\frac{\lambda}{c_n}|\phi|_{j,n}\Big) + \limsup_{n\to\infty} \frac{1}{n}\sum_{j\in\mathbb{Z}} \Lambda\Big(-\frac{\lambda}{c_n}|\phi|_{j,n}\Big). \end{aligned}$$

Keeping $\lambda > 0$ small, using Lemma 3.5 and Lemma 3.6 and then letting $x \to 0$ proves the limit as in (3.2) for the first integral under the probability, and the second and the third integrals are even simpler. The proof of (3.3) is similar, starting with

$$P\Big(\sup_{0\leq t\leq 1} |Y_n(t)| > M\Big) \leq P\Big(\frac{1}{a_n}\sum_{i=1}^n |X_i| > M\Big).$$

Now one establishes (3.2) for p by writing, for $M > 0$,

$$\begin{aligned} &P\Big[\int_0^{1-x} \Big|Y_n(t+x) - Y_n(t)\Big|^p dt + \int_0^x \Big|Y_n(t)\Big|^p dt + \int_{1-x}^1 \Big|Y_n(t)\Big|^p dt > \epsilon\Big] \\ \leq\ & P\Big[\int_0^{1-x} \Big|Y_n(t+x) - Y_n(t)\Big| dt + \int_0^x \Big|Y_n(t)\Big| dt + \int_{1-x}^1 \Big|Y_n(t)\Big| dt > \frac{\epsilon}{2M^{p-1}}\Big] \\ & + P\Big[\sup_{0\leq t\leq 1} |Y_n(t)| > M\Big], \end{aligned}$$

and letting first $n \to \infty$, $x \downarrow 0$, and then $M \uparrow \infty$. □

Lemma 3.4. *Under the assumptions* $S1$ *or* $R1$*, the correspondent upper rate functions* G^{sl} *in* (2.8) *and* G^{rl} *in* (2.16)*, are infinite outisde of the space* $\mathcal{BV}$*.*

Proof. Let $f \notin \mathcal{BV}$. Choose $\delta > 0$ so small that any λ with $|\lambda| \leq \delta$ is in $\mathcal{F}_\Lambda^\circ$ and a vector with k identical components $\lambda, \ldots, \lambda$ is in the interiors of both $\Pi_{t_1,\ldots,t_k}$ in (2.7) and $\Pi^{r,\alpha}_{t_1,\ldots,t_k}$ in (2.11) and (2.12). For $M > 0$ choose a partition $0 < t_1 < \cdots < t_k = 1$ of $[0,1]$ such that $\sum_{i=1}^k \big|f(t_i) - f(t_{i-1})\big| > M$. For $i = 1, \ldots, k$ such that $f(t_i) - f(t_{i-1}) \neq 0$ choose λ_i of length δ in the direction of $f(t_i) - f(t_{i-1})$. Then under, say, assumption $S1$,

$$\begin{aligned} G^{sl}(f) &\geq \sup_{\underline{\lambda}\in\Pi_{t_1,\ldots,t_k}} \sum_{i=1}^k \Big\{\lambda_i \cdot \big(f(t_i) - f(t_{i-1})\big) - (t_i - t_{i-1})\Lambda(\lambda_i)\Big\} \\ &\geq \delta M - \sup_{|\lambda|\leq\delta} \Lambda(\lambda). \end{aligned}$$

Letting $M \to \infty$ proves the statement under the assumption $S1$, and the argument under the assumption $R1$ is similar. □

Lemma 3.5. *Suppose $\Lambda : \mathbb{R}^d \to \mathbb{R}$ is the log-moment generating function of a mean zero random variable Z, with $0 \in \mathcal{F}_\Lambda^\circ$, $\sum_{i=-\infty}^{\infty} |\phi_i| < \infty$ with $\sum_{i=-\infty}^{\infty} \phi_i = 1$ and $0 < t_1 < \cdots < t_k \leq 1$.*

(i) For all $\underline{\lambda} = (\lambda_1, \ldots, \lambda_k) \in \Pi_{t_1,\ldots,t_k} \subset (\mathbb{R}^d)^k$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{l=-\infty}^{\infty} \Lambda\Big(\sum_{i=1}^{k} \lambda_i \phi_{l+[nt_{i-1}],[nt_i]-[nt_{i-1}]} \Big) = \sum_{i=1}^{k} (t_i - t_{i-1}) \Lambda(\lambda_i).$$

(ii) If $a_n/\sqrt{n} \to \infty$ and $a_n/n \to 0$ then for all $\underline{\lambda} \in (\mathbb{R}^d)^k$,

$$\lim_{n\to\infty} \frac{n}{a_n^2} \sum_{l=-\infty}^{\infty} \Lambda\Big(\frac{a_n}{n} \sum_{i=1}^{k} \lambda_i \phi_{l+[nt_{i-1}],[nt_i]-[nt_{i-1}]} \Big) = \sum_{i=1}^{k} (t_i - t_{i-1}) \lambda_i \cdot \Sigma \lambda_i,$$

where Σ is the covaraince matrix of Z.

(iii) If $\Lambda(\cdot)$ is balanced regular varying at ∞ with exponent $\beta > 1$, $a_n/n \to \infty$ and b_n is as defined as defined in assumption S4, then for all $\underline{\lambda} \in (\mathbb{R}^d)^k$,

$$\lim_{n\to\infty} \frac{1}{b_n} \sum_{l=-\infty}^{\infty} \Lambda\Big(\frac{b_n}{a_n} \sum_{i=1}^{k} \lambda_i \phi_{l+[nt_{i-1}],[nt_i]-[nt_{i-1}]} \Big)$$
$$= \sum_{i=1}^{k} (t_i - t_{i-1}) \zeta\Big(\frac{\lambda_i}{|\lambda_i|} \Big) |\lambda_i|^\beta.$$

Proof. (i) We begin by making a few observations:

(a) For every $\delta > 0$ there exists N_δ such that for all $n > N_\delta$

$$\sum_{|i| > (n \min_j (t_j - t_{j-1}))^{1/2}} |\phi_i| < \delta. \tag{3.4}$$

(b) For fixed $\underline{\lambda} = (\lambda_1, \ldots, \lambda_k) \in \Pi_{t_1,\ldots,t_k}$, there exists $M > 0$ such that for all $l \in \mathbb{Z}$ and all n large enough

$$\Big| \Lambda\Big(\sum_{i=1}^{k} \lambda_i \phi_{l+[nt_{i-1}],s_i} \Big) \Big| \leq M, \tag{3.5}$$

where $s_i = s_i(n) = [nt_i] - [nt_{i-1}]$. Since the zero mean of Z means that $\Lambda(x) = o(|x|)$ as $|x| \to 0$, it follows from (3.5) that there exists $C > 0$ such that in the same range of n and for all $l \in \mathbb{Z}$

$$\Big| \Lambda\Big(\sum_{i=1}^{k} \lambda_i \phi_{l+[nt_{i-1}],s_i} \Big) \Big| \leq C \Big| \sum_{i=1}^{k} \lambda_i \phi_{l+[nt_{i-1}],s_i} \Big|. \tag{3.6}$$

Let $L = \Big(|\lambda_1| + \cdots + |\lambda_k|\Big)$. Since Λ is continuous at λ_j, given $\epsilon > 0$ we can choose $\delta > 0$ so that for n large enough,

$$\Big|\sum_{i=1}^{k} \lambda_i \phi_{l+[nt_{i-1}],s_i} - \lambda_j\Big| < \delta,$$

for all $-[nt_j] + \sqrt{s_j} < l < -[nt_{j-1}] - \sqrt{s_j}$, and then

$$\Big|\frac{1}{n}\sum_{l=-[nt_j]+\sqrt{s_j}}^{-[nt_{j-1}]-\sqrt{s_j}} \Lambda\Big(\sum_{i=1}^{k} \lambda_i \phi_{l+[nt_{i-1}],s_i}\Big) - \frac{s_j - 2\sqrt{s_j}}{n}\Lambda(\lambda_j)\Big| < \epsilon.$$

Therefore for $j = 1, \dots, k$

$$\lim_{n\to\infty} \frac{1}{n}\sum_{l=-[nt_j]+\sqrt{s_j}}^{-[nt_{j-1}]-\sqrt{s_j}} \Lambda\Big(\sum_{i=1}^{k} \lambda_i \phi_{l+[nt_{i-1}],s_i}\Big) = (t_j - t_{j-1})\Lambda(\lambda_j). \tag{3.7}$$

Note that

$$\Big|\frac{1}{n}\sum_{l=-[nt_j]-\sqrt{s_j}}^{-[nt_j]+\sqrt{s_{j+1}}} \Lambda\Big(\sum_{i=1}^{k} \lambda_i \phi_{l+[nt_{i-1}],s_i}\Big)\Big| \overset{(3.5)}{\le} \frac{\sqrt{s_j} + \sqrt{s_{j+1}}}{n} M \overset{n\to\infty}{\longrightarrow} 0\,. \tag{3.8}$$

Finally, observe that for large n,

$$\begin{aligned} &\Big|\frac{1}{n}\sum_{l=-\infty}^{-[nt_k]-\sqrt{s_k}} \Lambda\Big(\sum_{i=1}^{k} \lambda_i \phi_{l+[nt_{i-1}],s_i}\Big)\Big| \\ \overset{(3.6)}{\le}\ & C\frac{1}{n}\sum_{l=-\infty}^{-[nt_k]-\sqrt{s_k}} \Big|\sum_{i=1}^{k} \lambda_i \phi_{l+[nt_{i-1}],s_i}\Big| \\ \le\ & CL\sum_{l=-\infty}^{-\sqrt{s_k}} |\phi_l| \overset{(i)}{\to} 0. \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} &\Big|\frac{1}{n}\sum_{l=\sqrt{s_1}}^{\infty} \Lambda\Big(\sum_{i=1}^{k} \lambda_i \phi_{l+[nt_{i-1}],s_i}\Big)\Big| \\ \overset{(3.6)}{\le}\ & C\frac{1}{n}\sum_{l=\sqrt{s_1}}^{\infty} \Big|\sum_{i=1}^{k} \lambda_i \phi_{l+[nt_{i-1}],s_i}\Big| \\ \le\ & CL\sum_{l=\sqrt{s_1}}^{\infty} |\phi_l| \to 0. \end{aligned} \tag{3.10}$$

Thus, combining (3.7), (3.8), (3.9) and (3.10) we get

$$\lim_{n\to\infty} \frac{1}{n} \sum_{l=-\infty}^{\infty} \Lambda\Big(\sum_{i=1}^{k} \lambda_i \phi_{l+[nt_{i-1}],[nt_i]-[nt_{i-1}]} \Big) = \sum_{i=1}^{k} (t_i - t_{i-1}) \Lambda(\lambda_i).$$

(ii) Since $\Lambda(x) \sim x \cdot \Sigma x/2$ as $|x| \to 0$, we get that for every $1 \le j \le k$,

$$\lim_{n\to\infty} \frac{n}{a_n^2} \sum_{l=-[nt_j]+\sqrt{s_j}}^{-[nt_{j-1}]-\sqrt{s_j}} \Lambda\Big(\frac{a_n}{n} \sum_{i=1}^{k} \lambda_i \phi_{l+[nt_{i-1}],[nt_i]-[nt_{i-1}]} \Big) = (t_j - t_{j-1}) \frac{1}{2} \lambda_j \cdot \Sigma \lambda_j.$$

The rest of the proof is similar to the proof of part (i).

(iii) Since $\Lambda(\lambda)$ is regular varying at infinity with exponent $\beta > 1$, we get that for every $1 \le j \le k$,

$$\lim_{n\to\infty} \frac{1}{b_n} \sum_{l=-[nt_j]+\sqrt{s_j}}^{-[nt_{j-1}]-\sqrt{s_j}} \Lambda\Big(\frac{b_n}{a_n} \sum_{i=1}^{k} \lambda_i \phi_{l+[nt_{i-1}],[nt_i]-[nt_{i-1}]} \Big)$$

$$= (t_j - t_{j-1}) \zeta\Big(\frac{\lambda_j}{|\lambda_j|} \Big) |\lambda_j|^{\beta}.$$

The rest of the proof is, once again, similar to the proof of part (i). □

Lemma 3.6. *Suppose $\Lambda : \mathbb{R}^d \to \mathbb{R}$ is the log-moment generating function of a mean zero random variable, with $0 \in \mathcal{F}_\Lambda^\circ$, the coefficients of the moving average are balanced regularly varying with exponent α as in Assumption 2.3, and $0 < t_1 < \cdots < t_k \le 1$.*

(i) For all $\underline{\lambda} = (\lambda_1, \ldots, \lambda_k) \in \Pi^{r,\alpha}_{t_1,\ldots,t_k} \subset (\mathbb{R}^d)^k$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{l=-\infty}^{\infty} \Lambda\Big(\frac{1}{\Psi_n} \sum_{i=1}^{k} \lambda_i \phi_{l+[nt_{i-1}],[nt_i]-[nt_{i-1}]} \Big) = \Lambda^{rl}_{t_1,\cdots,t_k}(\underline{\lambda}).$$

(ii) If $a_n/\sqrt{n} \to \infty$ and $a_n/n \to 0$ then for all $\underline{\lambda} \in (\mathbb{R}^d)^k$,

$$\lim_{n\to\infty} \frac{n\Psi_n^2}{a_n^2} \sum_{l=-\infty}^{\infty} \Lambda\Big(\frac{a_n}{n\Psi_n^2} \sum_{i=1}^{k} \lambda_i \phi_{l+[nt_{i-1}],[nt_i]-[nt_{i-1}]} \Big)$$

$$= \begin{cases} \int_{-\infty}^{\infty} G_\Sigma\Big(h_{t_1,\ldots,t_k}(x;\underline{\lambda}) \Big) dx & if \quad \alpha < 1 \\ \sum_{i=1}^{k} (t_i - t_{i-1}) G_\Sigma(\lambda_i) & if \quad \alpha = 1, \end{cases}.$$

(iii) If $a_n/n \to \infty$, b_n is as defined in assumption R4, and $\Lambda(\cdot)$ is balanced regular varying at ∞ with exponent $\beta > 1$, then for all $\underline{\lambda} \in (\mathbb{R}^d)^k$,

$$\lim_{n\to\infty} \frac{1}{b_n} \sum_{l=-\infty}^{\infty} \Lambda\Big(\frac{b_n}{a_n} \sum_{i=1}^{k} \lambda_i \phi_{l+[nt_{i-1}],[nt_i]-[nt_{i-1}]}\Big)$$

$$= \begin{cases} \int\limits_{-\infty}^{\infty} \Lambda^h\Big(h_{t_1,\dots,t_k}(x;\underline{\lambda})\Big)dx & if \quad \alpha < 1 \\ \sum\limits_{i=1}^{k}(t_i - t_{i-1})\Lambda^h(\lambda_i) & if \quad \alpha = 1, \end{cases} .$$

Proof. (i) We may (and will) assume that $t_k = 1$, since we can always add an extra point with the zero vector λ corresponding to it. Let us first assume that $\alpha < 1$. Note that for any $m \geq 1$ and large n,

$$\begin{aligned} & \frac{1}{n} \sum_{j=nm+1}^{n(m+1)} \Lambda\Big(\frac{1}{\Psi_n} \sum_{i=1}^{k} \lambda_i \phi_{j+[nt_{i-1}],[nt_i]-[nt_{i-1}]}\Big) \\ = \; & \frac{1}{n} \sum_{j=nm+1}^{n(m+1)} \Lambda\Big(\sum_{i=1}^{k} \lambda_i \frac{n\psi(n)}{\Psi_n} \frac{1}{n}\big(\frac{\phi_{j+[nt_{i-1}]+1}}{\psi(n)} + \cdots + \frac{\phi_{j+[nt_i]}}{\psi(n)}\big)\Big) \\ = \; & \int_m^{m+1} f_n(x)\, dx\,, \end{aligned}$$

where

$$f_n(x) = \Lambda\Big(\frac{1}{\Psi_n} \sum_{i=1}^{k} \lambda_i \phi_{j+[nt_{i-1}],[nt_i]-[nt_{i-1}]}\Big)$$

if $(j-1)/n < x \leq j/n$ for $j = nm+1, \dots, n(m+1)$.

Notice that by Karamata's theorem (see Resnick (1987)), $n\psi(n)/\Psi_n \to 1-\alpha$ as $n \to \infty$. Furthermore, given $0 < \epsilon < \alpha$, we can use Potter's bounds (see Proposition 0.8 *ibid*) to check that there is n_ϵ such that for all $n \geq n_\epsilon$, for all $k = [nt_{i-1}]+1, \dots, [nt_i]$, $m-1 < x \leq m$ and $(j-1)/n < x \leq j/n$

$$\frac{\phi_{j+k}}{\psi(n)} = \frac{\phi_{j+k}}{\psi(j+k)} \frac{\psi(j+k)}{\psi(j)} \frac{\psi(j)}{\psi(n)}$$

$$\in \left((1-\epsilon)\, p \left(\frac{j+k}{j}\right)^{-(\alpha+\epsilon)} x^{-\alpha},\, (1+\epsilon)\, p \left(\frac{j+k}{j}\right)^{-(\alpha-\epsilon)} x^{-\alpha}\right),$$

and so for n large enough,

$$\frac{1}{n}\big(\frac{\phi_{j+[nt_{i-1}]+1}}{\psi(n)} + \cdots + \frac{\phi_{j+[nt_i]}}{\psi(n)}\big) \tag{3.11}$$

$$\in \left((1-\epsilon)\, p \int_{t_{i-1}}^{t_i} \left(\frac{y+x}{x}\right)^{-(\alpha+\epsilon)} x^{-\alpha}\, dy,\, (1+\epsilon)\, p \int_{t_{i-1}}^{t_i} \left(\frac{y+x}{x}\right)^{-(\alpha-\epsilon)} x^{-\alpha}\, dy\right).$$

Therefore,

$$\frac{1}{\Psi_n}\sum_{i=1}^{k}\lambda_i\phi_{j+[nt_{i-1}],[nt_i]-[nt_{i-1}]} \to (1-\alpha)\,p\sum_{i=1}^{k}\lambda_i\int_{t_{i-1}}^{t_i}(y+x)^{-\alpha}\,dy$$

$$= p\sum_{i=1}^{k}\lambda_i\Big((t_i+x)^{1-\alpha}-(t_{i-1}+x)^{1-\alpha}\Big).$$

This last vector is a convex linear combination of the vectors $p\big((1+x)^{1-\alpha}-x^{1-\alpha}\big)\lambda_i,\ i=1\ldots,k$. By the definition of the set $\Pi^{r,\alpha}_{t_1,\ldots,t_k}$, each one of these vectors belongs to $\mathcal{F}^\circ_\Lambda$ and, by convexity of Λ, so does the convex linear combination. Therefore,

$$\Lambda\Big(\frac{1}{\Psi_n}\sum_{i=1}^{k}\lambda_i\phi_{j+[nt_{i-1}],[nt_i]-[nt_{i-1}]}\Big) \to \Lambda\Big(p\sum_{i=1}^{k}\lambda_i\Big((t_i+x)^{1-\alpha}-(t_{i-1}+x)^{1-\alpha}\Big)\Big).$$

This convexity argument also shows that the function f_n is uniformly bounded on $(m,m+1]$ for large enough n, and so we conclude that for any $m\geq 1$

$$\frac{1}{n}\sum_{j=nm+1}^{n(m+1)}\Lambda\Big(\frac{1}{\Psi_n}\sum_{i=1}^{k}\lambda_i\phi_{j+[nt_{i-1}],[nt_i]-[nt_{i-1}]}\Big)$$

$$\to \int_m^{m+1}\Lambda\Big((1-\alpha)\sum_{i=1}^{k}\lambda_i\int\limits_{x+t_{i-1}}^{x+t_i}py^{-\alpha}dy\Big)dx.$$

Similar arguments show that for $m\leq -3$

$$\frac{1}{n}\sum_{j=nm+1}^{n(m+1)}\Lambda\Big(\frac{1}{\Psi_n}\sum_{i=1}^{k}\lambda_i\phi_{j+[nt_{i-1}],[nt_i]-[nt_{i-1}]}\Big)$$

$$\to \int_m^{m+1}\Lambda\Big((1-\alpha)\sum_{i=1}^{k}\lambda_i\int\limits_{x+t_{i-1}}^{x+t_i}q|y|^{-\alpha}dy\Big)dx,$$

and that for any $\delta>0$,

$$\frac{1}{n}\sum_{j=-2n+1}^{-n-n\delta}\Lambda\Big(\frac{1}{\Psi_n}\sum_{i=1}^{k}\lambda_i\phi_{j+[nt_{i-1}],[nt_i]-[nt_{i-1}]}\Big)$$

$$\to \int_{-2}^{-1-\delta}\Lambda\Big((1-\alpha)\sum_{i=1}^{k}\lambda_i\int\limits_{x+t_{i-1}}^{x+t_i}q|y|^{-\alpha}dy\Big)dx$$

and

$$\frac{1}{n}\sum_{j=n\delta}^{n}\Lambda\Big(\frac{1}{\Psi_n}\sum_{i=1}^{k}\lambda_i\phi_{j+[nt_{i-1}],[nt_i]-[nt_{i-1}]}\Big)$$

$$\to\int_\delta^1\Lambda\Big((1-\alpha)\sum_{i=1}^{k}\lambda_i\int\limits_{x+t_{i-1}}^{x+t_i}py^{-\alpha}dy\Big)dx.$$

Using once again the same argument we see that for small δ

$$\frac{1}{n}\sum_{j=-n}^{0}\mathbf{1}\Big(\Big|\frac{j}{n}+t_i\Big|>\delta\text{ all }i=1,\ldots,k\Big)\Lambda\Big(\frac{1}{\Psi_n}\sum_{i=1}^{k}\lambda_i\phi_{j+[nt_{i-1}],[nt_i]-[nt_{i-1}]}\Big)$$

$$\to\int_{-1}^{0}\mathbf{1}\Big(|x+t_i|>\delta\text{ all }i=1,\ldots,k\Big)$$

$$\Lambda\Big((1-\alpha)\sum_{i=1}^{k}\lambda_i\int\limits_{x+t_{i-1}}^{x+t_i}|y|^{-\alpha}\big(pI_{[y\geq 0]}+qI_{[y<0]}\big)dy\Big)dx.$$

We have covered above all choices of the subscript j apart from a finite number of stretches of j of length at most $n\delta$ each. By the definition of the set $\Pi^{r,\alpha}_{t_1,\ldots,t_k}$ we see that there is a finite K such that for all n large enough,

$$\frac{1}{n}\sum_{j\text{ not yet considered}}\Lambda\Big(\frac{1}{\Psi_n}\sum_{i=1}^{k}\lambda_i\phi_{j+[nt_{i-1}],[nt_i]-[nt_{i-1}]}\Big)\leq K\delta.$$

It follows from (3.11) and the fact that $\Lambda(\lambda)=O(|\lambda|^2)$ as $\lambda\to 0$ that for all $|m|$ large enough there is $C\in(0,\infty)$ such that

$$\frac{1}{n}\sum_{nm+1}^{n(m+1)}\Lambda\Big(\frac{1}{\Psi_n}\sum_{i=1}^{k}\lambda_i\phi_{j+[nt_{i-1}],[nt_i]-[nt_{i-1}]}\Big)\leq C|m|^{-2\alpha}$$

for all n large enough. This is summable by the assumption on α, and so the dominated convergence theorem gives us the result.

Next we move our attention to the case when $\alpha=1$. Choose any $\delta>0$. By the slow variation of Ψ_n we see that

$$\sup_{j>\delta n\text{ or }j<-(1+\delta)n}\frac{|\phi_{j,n}|}{\Psi_n}\to 0\,,$$

while for any $0<x<1$ we have

$$\frac{\phi_{0,[nx]}}{\Psi_n}\to p\text{ and }\frac{\phi_{-[nx],[nx]}}{\Psi_n}\to q.$$

Write

$$\frac{1}{n}\sum_{j=-n+1}^{0}\Lambda\Big(\frac{1}{\Psi_n}\sum_{i=1}^{k}\lambda_i\phi_{j+[nt_{i-1}],[nt_i]-[nt_{i-1}]}\Big)$$

$$=\sum_{m=1}^{k}\frac{1}{n}\sum_{j=-[nt_m]+1}^{j=-[nt_{m-1}]}\Lambda\Big(\sum_{i=1}^{k}\lambda_i\frac{\phi_{j+[nt_{i-1}],[nt_i]-[nt_{i-1}]}}{\Psi_n}\Big)\,.$$

Fix $m=1,\dots,k$, and observe that, for any $\epsilon>0$ and n large enough,

$$\frac{1}{n}\sum_{j=-[nt_m]+1}^{-[nt_{m-1}]}\Lambda\Big(\sum_{i=1}^{k}\lambda_i\frac{\phi_{j+[nt_{i-1}],[nt_i]-[nt_{i-1}]}}{\Psi_n}\Big)=\int_{-t_m-\epsilon}^{-t_{m-1}}f_n(x)\,dx\,,$$

where this time

$$f_n(x)=\mathbf{1}\Big(-\frac{[nt_m]}{n}<x\le-\frac{[nt_{m-1}]}{n}\Big)\Lambda\Big(\sum_{i=1}^{k}\lambda_i\frac{\phi_{j+[nt_{i-1}],[nt_i]-[nt_{i-1}]}}{\Psi_n}\Big)$$

if $(j-1)/n<x\le j/n$ for $j=-[nt_m]+1,\dots,-[nt_{m-1}]$, otherwise $f_n(x)=0$. Clearly, $f_n(x)\to 0$ as $n\to\infty$ for all $-t_m-\epsilon<x<-t_m$. Furthermore,

$$\frac{\phi_{j+[nt_{i-1}],[nt_i]-[nt_{i-1}]}}{\Psi_n}\to 0$$

uniformly in $i\ne m$ and $j=-[nt_m]+1,\dots,-[nt_{m-1}]$, while for every $-t_m<x<-t_{m-1}$,

$$\frac{\phi_{j+[nt_{m-1}],[nt_i]-[nt_{m-1}]}}{\Psi_n}\to p+q=1\,.$$

By the definition of the set $\Pi^{r,1}_{t_1,\dots,t_k}$ we see that $f_n\to\mathbf{1}_{(-t_m,-t_{m-1})}\Lambda(\lambda_m)$ a.e., and that the functions f_n are uniformly bounded for large n. Therefore,

$$\frac{1}{n}\sum_{j=-n+1}^{0}\Lambda\Big(\frac{1}{\Psi_n}\sum_{i=1}^{k}\lambda_i\phi_{j+[nt_{i-1}],[nt_i]-[nt_{i-1}]}\Big)\to\sum_{m=1}^{k}(t_m-t_{m-1})\Lambda(\lambda_m)\,.$$

Finally, the argument above, using Potter's bounds and the fact that $\Lambda(\lambda)=O(|\lambda|^2)$ as $\lambda\to 0$, shows that

$$\frac{1}{n}\sum_{j\notin[-n,0]}\Lambda\Big(\frac{1}{\Psi_n}\sum_{i=1}^{k}\lambda_i\phi_{j+[nt_{i-1}],[nt_i]-[nt_{i-1}]}\Big)\to 0.$$

This completes the proof of part (i).

For part (ii) consider, once again, the cases $1/2<\alpha<1$ and $\alpha=1$ separately. If $1/2<\alpha<1$, then for every $m\ge 1$ we use the regular variation and the fact that $\Lambda(x)\sim x\cdot\Sigma x/2$ as $|x|\to 0$,

$$\frac{n\Psi_n^2}{a_n^2}\sum_{j=nm+1}^{n(m+1)}\Lambda\Big(\frac{a_n}{n\Psi_n^2}\sum_{i=1}^{k}\lambda_i\phi_{l+[nt_{i-1}],[nt_i]-[nt_{i-1}]}\Big)\to$$

$$\int_m^{m+1} \Big((1-\alpha)\sum_{i=1}^k \lambda_i \int_{x+t_{i-1}}^{x+t_i} py^{-\alpha}dy\Big)\cdot\Sigma\Big((1-\alpha)\sum_{i=1}^k \lambda_i \int_{x+t_{i-1}}^{x+t_i} py^{-\alpha}dy\Big)/2\,dx\,,$$

and we proceed as in the proof of part (i), considering the various other ranges of m, obtaining the result. If $\alpha = 1$, then for any $m = 1,\ldots,k$, by the regular variation and the fact that $\Lambda(x) \sim x\cdot\Sigma x/2$ as $|x|\to 0$, one has

$$\frac{n\Psi_n^2}{a_n^2}\sum_{j=-[nt_m]+1}^{[nt_{m-1}]} \Lambda\Big(\frac{a_n}{n\Psi_n}\sum_{i=1}^k \lambda_i \frac{\phi_{j+[nt_{i-1}],[nt_i]-[nt_{i-1}]}}{\Psi_n}\Big) \to \int_{-t_m}^{-t_{m-1}} \frac{1}{2}\lambda_m\cdot\Sigma\lambda_m\,dx\,,$$

and so

$$\frac{n\Psi_n^2}{a_n^2}\sum_{j=-n+1}^{0} \Lambda\Big(\frac{a_n}{n\Psi_n^2}\sum_{i=1}^k \lambda_i\phi_{j+[nt_{i-1}],[nt_i]-[nt_{i-1}]}\Big) \to \frac{1}{2}\sum_{m=1}^k (t_m-t_{m-1})\lambda_m\cdot\Sigma\lambda_m\,.$$

As in part (i), by using Potter's bounds and the fact that $\Lambda(\lambda) = O(|\lambda|^2)$ as $\lambda\to 0$, shows that

$$\frac{n\Psi_n^2}{a_n^2}\sum_{j\notin[-n,0]} \Lambda\Big(\frac{a_n}{n\Psi_n^2}\sum_{i=1}^k \lambda_i\phi_{j+[nt_{i-1}],[nt_i]-[nt_{i-1}]}\Big) \to 0,$$

giving us the desired result.

We proceed in a similar fasion in part (iii). If $1/2<\alpha<1$, then, for example, for $m\geq 1$, by the regular variation at infinity,

$$\frac{1}{b_n}\sum_{j=nm+1}^{n(m+1)} \Lambda\Big(\frac{b_n}{a_n}\sum_{i=1}^k \lambda_i\phi_{l+[nt_{i-1}],[nt_i]-[nt_{i-1}]}\Big) \to$$

$$\int_m^{m+1} \zeta\left(\frac{(1-\alpha)\sum_{i=1}^k \lambda_i \int_{x+t_{i-1}}^{x+t_i} py^{-\alpha}dy}{\Big|(1-\alpha)\sum_{i=1}^k \lambda_i \int_{x+t_{i-1}}^{x+t_i} py^{-\alpha}dy\Big|}\right)\Big|(1-\alpha)\sum_{i=1}^k \lambda_i \int_{x+t_{i-1}}^{x+t_i} py^{-\alpha}dy\Big|^\beta$$

(if the argument of the function ζ is $0/0$, then the integrand is set to be equal to zero), and we treat the other ranges of m in a similar to way to what has been done in part (ii). This gives us the stated limit. For $\alpha = 1$ we have for any $m = 1,\ldots,k$, by the regular variation at infinity,

$$\frac{1}{b_n}\sum_{j=-[nt_m]+1}^{[nt_{m-1}]} \Lambda\Big(\frac{b_n}{a_n}\sum_{i=1}^k \lambda_i\phi_{j+[nt_{i-1}],[nt_i]-[nt_{i-1}]}\Big) \to \int_{-t_m}^{-t_{m-1}} \zeta\left(\frac{\lambda_m}{|\lambda_m|}\right)|\lambda_m|^\beta\,dx\,,$$

and so

$$\frac{1}{b_n}\sum_{j=-n+1}^{0} \Lambda\Big(\frac{b_n}{a_n}\sum_{i=1}^k \lambda_i\phi_{j+[nt_{i-1}],[nt_i]-[nt_{i-1}]}\Big) \to \sum_{m=1}^k (t_m-t_{m-1})\zeta\left(\frac{\lambda_m}{|\lambda_m|}\right)|\lambda_m|^\beta\,,$$

while the sum over the rest of the range of j contributes only terms of a smaller order. Hence the result. $\square$

Remark 3.7. The argument in the proof shows also that the statements of all three parts of the lemma remain true if the sums $\sum_{l=-\infty}^{\infty}$ are replaced by sums $\sum_{l=-A_n}^{A_n}$ with $n/A_n \to 0$ as $n \to \infty$.

Lemma 3.8. *For $1/2 < \alpha < 1$, let $h_{t_1,\ldots,t_k}$ be defined by (2.13), and $\Lambda^{rl}_{t_1,\cdots,t_k}$ defined by (2.17). Then for any function of bounded variation f on $[0,1]$ satisfying $f(0) = 0$,*

$$\sup_{j\in J}(\Lambda^{rl}_{t_1,\ldots,t_{|j|}})^*\big(f(t_1), f(t_2)-f(t_1),\ldots,f(t_{|j|})-f(t_{|j|-1})\big)$$

$$= \begin{cases} \Lambda^*_\alpha(f') & if \quad f \in \mathcal{AC}, \\ \infty & otherwise, \end{cases}$$

where Λ^*_α is defined by (2.14).

Proof. First assume that $f \in \mathcal{AC}$. It is easy to see that the inequality $\Lambda^*_\alpha(f') \geq \sup_{j\in J}(\Lambda^{rl}_{t_1,\cdots,t_{|j|}})^*(f(t_1), f(t_2)-f(t_1),\ldots,f(t_{|j|}))$ holds by considering a function $\psi \in L_\infty[0,1]$, which takes the value λ_i in the interval $(t_{i-1}, t_i]$. For the other inequality, we start by observing that the supremum in the definition of Λ^*_α in (2.14) is achieved over those $\psi \in L_\infty[0,1]$ such that, for almost all real x, the integral

$$I_x = \int_0^1 \psi(t)(1-\alpha)|x+t|^{-\alpha}\Big[pI_{[x+t\geq 0]} + qI_{[x+t<0]}\Big]\,dt \in \mathcal{F}_\Lambda\,,$$

and, hence, also over those $\psi \in L_\infty[0,1]$ such that $I_x \in \mathcal{F}^\circ_\Lambda$ for almost every x.

For any ψ as above choose a sequence of uniformly bounded functions ψ^n converging to ψ, almost everywhere on $[0,1]$, such that for every n, ψ^n is of the form $\sum_i \lambda_i^n I_{A_i^n}$, where $A_i^n = (t_{i-1}^n, t_i^n]$, for some $0 < t_1^n < t_2^n < \cdots < t_{k_n}^n = 1$. Then by the continuity of Λ over $\mathcal{F}^\circ_\Lambda$ and Fatou's lemma,

$$\int_0^1 \psi(t)f'(t)dt - \int_{-\infty}^{\infty}\Lambda\left(\int_0^1 \psi(t)(1-\alpha)|x+t|^{-\alpha}\Big[pI_{[x+t\geq 0]} + qI_{[x+t<0]}\Big]\,dt\right)dx$$

$$\begin{aligned} = \quad & \int_0^1 \lim_n \psi^n(t)f'(t)dt \\ & - \int_{-\infty}^{\infty}\Lambda\left(\int_0^1 \lim_n \psi^n(t)(1-\alpha)|x+t|^{-\alpha}\Big[pI_{[x+t\geq 0]} + qI_{[x+t<0]}\Big]\,dt\right)dx \\ = \quad & \lim_n \int_0^1 \psi^n(t)f'(t)dt \\ & - \int_{-\infty}^{\infty}\lim_n \Lambda\left(\int_0^1 \psi^n(t)(1-\alpha)|x+t|^{-\alpha}\Big[pI_{[x+t\geq 0]} + qI_{[x+t<0]}\Big]\,dt\right)dx \end{aligned}$$

$$
\begin{aligned}
&\leq \lim_n \int_0^1 \psi^n(t) f'(t) dt \\
&\quad - \limsup_n \int_{-\infty}^{\infty} \Lambda \left(\int_0^1 \psi^n(t)(1-\alpha)|x+t|^{-\alpha} \left[p I_{[x+t\geq 0]} + q I_{[x+t<0]} \right] dt \right) dx \\
&= \liminf_n \left\{ \sum_{i=1}^{k_n} \lambda_i^n \cdot \big(f(t_i^n) - f(t_{i-1}^n) \big) - \Lambda^{rl}_{t_1^n,\cdots,t_n^n}(\lambda_1^n, \cdots, \lambda_n^n) \right\} \\
&\leq \sup_{j\in J} (\Lambda^{rl}_{t_1,\cdots,t_{|j|}})^* \big(f(t_1), f(t_2) - f(t_1), \ldots, f(t_{|j|}) - f(t_{|j|-1}) \big).
\end{aligned}
$$

Now suppose that f is not absolutely continuous. That is, there exists $\epsilon > 0$ and $0 \leq r_1^n < s_1^n \leq r_2^n < \cdots \leq r_{k_n}^n < s_{k_n}^n \leq 1$, such that $\sum_{i=1}^{k_n}(s_i^n - r_i^n) \to 0$ but $\sum_{i=1}^{k_n} |f(s_i^n) - f(r_i^n)| \geq \epsilon$. Let j^n be such that $t_{2p}^n = s_p^n$ and $t_{2p-1}^n = r_p^n$ (so that $|j^n| = 2k_n$). Now

$$
\begin{aligned}
&\sup_{j\in J} (\Lambda^{rl}_{t_1,\cdots,t_{|j|}})^* \big(f(t_1), f(t_2) - f(t_1), \ldots, f(t_{|j|}) - f(t_{|j|-1}) \big) \\
\geq\ & \limsup_n \left\{ \sup_{\underline{\lambda}^n \in \mathbb{R}^{2k_n}} \sum_{i=1}^{2k_n} \lambda_i^n \cdot \big(f(t_i^n) - f(t_{i-1}^n) \big) - \Lambda^{rl}_{t_1,\cdots,t_{2k_n}}(\underline{\lambda}^n) \right\} \\
\geq\ & \limsup_n \left\{ A \sum_{i=1}^{k_n} \big| f(s_i^n) - f(r_i^n) \big| - \Lambda^{rl}_{t_1,\cdots,t_{2k_n}}(\underline{\lambda}^{n*}) \right\} \geq A\epsilon,
\end{aligned}
$$

where $\lambda_{2p-1}^{n*} = 0$ and $\lambda_{2p}^{n*} = A\big(f(s_i^n) - f(r_i^n)\big)/|f(s_i^n) - f(r_i^n)|$ ($= 0$ if $f(s_i^n) - f(r_i^n) = 0$) for any $A > 0$. The last inequality follows from an application of dominated convergence theorem, quadratic behaviour of Λ at 0 and the fact that $h_{t_1,\cdots,t_{2k_n}}(x; \underline{\lambda}^{n*}) \to 0$ as $n \to \infty$, for every $x \in \mathbb{R}$. This completes the proof since A is arbitrary. □

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