ORIE 6334 Bridging Continuous and Discrete Optimization Sept. 18, 2019

Problem Set 1

Due Date: October 2, 2019

As a reminder, the collaboration policy from the syllabus is as follows:

> Your work on problem sets and exams should be your own. You may discuss approaches to problems with other students, but as a general guideline, such discussions may not involve taking notes. You must write up solutions on your own independently, and acknowledge anyone with whom you discussed the problem by writing their names on your problem set. You may **not** use papers or books or other sources (e.g. material from the web) to help obtain your solution.

1. Suppose that G is connected and Δ is the maximum degree of any node in G. Let λ_1 be the maximum eigenvalue of the adjacency matrix A of G. Prove that G is Δ-regular if and only if $\lambda_1 = \Delta$.

2. We can use the multiplicative weights algorithm to upper bound the costs we pay per time step rather than to lower bound the value we get per time step. Suppose that in time step t, if we make decision j, we pay a cost $c_t(j) \in [-1, 1]$. Show that if $\epsilon \leq 1/2$, then after T rounds, for any decision j, we have that the expected cost of our solution, $\sum_{t=1}^{T} \sum_{i=1}^{N} c_t(i) \cdot p_t(i)$, is at most
$$\sum_{t=1}^{T} c_t(j) + \epsilon \sum_{t=1}^{T} |c_t(j)| + \frac{1}{\epsilon} \ln N.$$
You may use the inequalities that $(1 - \epsilon x) \geq (1 - \epsilon)^x$ for $x \in [0, 1]$, $(1 - \epsilon x) \geq (1 + \epsilon)^{-x}$ for $x \in [-1, 0]$, and $\ln(1/(1 - \epsilon)) \leq \epsilon + \epsilon^2$ for $\epsilon \leq 1/2$.

3. In class, we proved that for $A \in \Re^{n \times n}$, A symmetric, (real) eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$, and orthonormal eigenvectors $x_1, \ldots, x_n$, that
$$\lambda_{k+1} = \min_{x \perp span(x_1, \ldots, x_k)} \frac{x^T A x}{x^T x}.$$
We asserted that it was also the case that
$$\begin{aligned} \lambda_{k+1} &= \min_{x \in span(x_{k+1}, \ldots, x_n)} \frac{x^T A x}{x^T x} \\ &= \max_{x \perp span(x_{k+2}, \ldots, x_n)} \frac{x^T A x}{x^T x} \\ &= \max_{x \in span(x_1, \ldots, x_{k+1})} \frac{x^T A x}{x^T x}. \end{aligned}$$
Prove that the first two equalities are true.

4. One problem with the characterization of eigenvalues we've seen so far is that it requires us to know the eigenvectors $x_1, \ldots, x_k$ (or $x_{k+2}, \ldots, x_n$) in order to compute λ_{k+1}. The *Courant-Fischer theorem* gives us a more general way of computing these eigenvalues. Prove that the following is true:

$$\begin{aligned}\lambda_{k+1} &= \min_{W \subseteq \Re^n : dim(W) = k+1} \max_{x \in W} \frac{x^T A x}{x^T x} \\ &= \max_{W \subseteq \Re^n : dim(W) = n-k} \min_{x \in W} \frac{x^T A x}{x^T x}.\end{aligned}$$