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Lecture 8

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1 The Matrix-Tree Theorem

In this lecture, we continue to see the usefulness of the graph Laplacian via its connection to yet another standard concept in graph theory, the spanning tree. Let $A[i]$ be the matrix A with its i^{th} row and column removed. We will give two different proofs of the following.

Theorem 1 (Kirchhoff's Matrix-Tree Theorem) *The number of spanning trees in a graph G is given by $\det(L_G[i])$, for any i.*

For the first proof, we will need the following fact.

Fact 1 *Let $A \in \mathbb{R}^{n\times n}$ and take E_{ii} to be the matrix with 1 in the $(i,i)^{th}$ entry and 0s elsewhere. Then,*

$$\det(A + E_{ii}) = \det(A) + \det(A[i]).$$

This claim is evident if one considers the permutation sum definition of the determinant

$$\det(A = (a_{ij})) = \sum_{\pi} \operatorname{sgn}(\pi)\, a_{1\pi(1)}a_{2\pi(2)} \dots a_{n\pi(n)},$$

since we've increased the $(i,i)^{\text{th}}$ entry, a_{ii}, to $a_{ii}+1$. Hence, we get the original determinant plus what is effectively a sum over all permutations of $[n] \setminus \{i\}$, avoiding the i^{th} row and column, i.e. $\det(A[i])$.

Proof of Theorem 1:

Our first proof will be by induction on the number of vertices and edges of the graph G.

__**Base case**__: If G is an empty graph on two vertices, then

$$L_G = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

so $L_G[i] = [0]$ and $\det(L_G[i]) = 0$, as desired.

__**Inductive step**__: In what follows, let $\tau(G)$ denote the number of spanning trees in G, let $G - e$ denote the graph with edge e removed, and let G/e denote the graph with edge e contracted. See below for an illustration of graph contraction.

If i is an isolated vertex, then G admits no spanning trees, and there are zeros along the i^{th} row and column of L_G. Thus, $\det(L_G[i]) = \det(L_{G-i}) = 0$, as desired.

[0]These notes are slightly modified from a Fall 2016 version scribed by Faisal Alkaabneh. This lecture is derived from Cvetković, Rowlinson, and Simić, *An Introduction to the Theory of Graph Spectra*, Sections 7.1 and 7.2, and Godsil and Royle, *Algebraic Graph Theory*, Section 13.2.

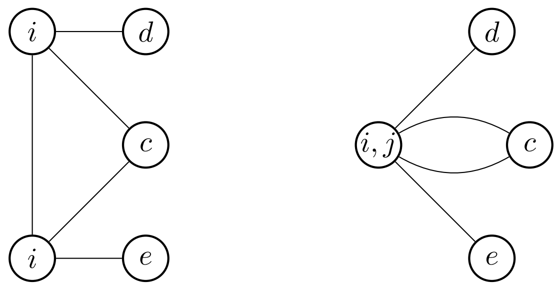

Figure 1: Example of contracting nodes i and j into a single node.

Otherwise, we can suppose that there exists an edge $e = (i, j)$ incident to i. For any spanning tree T, either $e \in T$ or $e \notin T$. We note that $\tau(G/e)$ gives the number of trees T with $e \in T$, while $\tau(G - e)$ gives the number of trees T with $e \notin T$. Thus,

$$\tau(G) = \tau(G \backslash e) + \tau(G - e).$$

Note that the first term is G with one fewer edge, while the second has one fewer vertex, so these will serve as the basis of our induction.

First we try to relate L_G to L_{G-e}, and we observe that $L_G[i] = L_{G-e}[i] + E_{jj}$ (that is, if we remove edge e, then the resulting Laplacian only differs from $L_G[i]$ in the degree of j). Thus, by Fact 1,

$$\begin{aligned}\det(L_G[i]) &= \det(L_{G-e} + E_{jj}) \\ &= \det(L_{G-e}[i]) + \det(L_{G-e}[i, j]) \\ &= \det(L_{G-e}[i]) + \det(L_G[i, j]),\end{aligned}$$

where by $L_G[i, j]$ we mean L_G with both the ith and jth rows and columns removed; the last equality follows since, once we've removed both the ith and jth rows and columns, there's no difference between L_G and L_{G-e} for $e = (i, j)$.

Now we will relate L_G to $L_{G/e}$. Suppose we contract i onto j (so that $L_{G/e}$ has no row or column corresponding to i). Then, $L_{G/e}[j] = L_G[i, j]$.

Thus, we have that

$$\begin{aligned}\det(L_G[i]) &= \det(L_{G-e}[i]) + \det(L_{G/e}[j]) \\ &= \tau(G - e) + \tau(G/e) = \tau(G).\end{aligned}$$

where the second equation follows by induction; this completes the proof. □

For the second proof of the theorem, we need the following fact which explains how to take the determinant of the product of rectangular matrices.

Fact 2 (Cauchy-Binet Formula) *Let $A \in \mathbb{R}^{n \times m}$, $B \in \mathbb{R}^{m \times n}$, with $m \geq n$. Let A_S (respectively B_S) be the submatrices formed by taking the columns (respectively rows) indexed by $S \subseteq [m]$ of A (respectively B).*

Let $\binom{[m]}{n}$ be the set of all size n subsets of $[m]$. Then,

$$\det(AB) = \sum_{S \in \binom{[m]}{n}} \det(A_S)\det(B_S).$$

Recall that $L_G = \sum_{(i,j)\in E}(e_i - e_j)(e_i - e_j)^T$. Thus, we can write $L_G = BB^T$ where $B \in \mathbb{R}^{m\times n}$ has one column per edge (i,j), with the column equal to $(e_i - e_j)$. (Since we can write $L_G = BB^T$, this is yet another proof that L_G is positive semidefinite.) Then, if $B[i]$ denotes B with its i^{i} row omitted, we have $L_G[i] = B[i]B[i]^T$. We let $B_S[i]$ denote $B[i]$ with just the columns indexed by $S \subseteq E$.

We also need the following lemma, whose proof we defer for a moment.

Lemma 2 *For $S \subseteq E$, $|S| = n-1$,*

$$|\det(B_S[i])| = \begin{cases} 1, & \textit{if } S \textit{ is a spanning tree} \\ 0, & \textit{otherwise.} \end{cases}$$

The second proof of the matrix-tree theorem now becomes very short.

Proof of Theorem 1:

$$\begin{aligned}\det(L_G[i]) &= \det(B[i]B[i]^T) \\ &= \sum_{S\in\binom{E}{n-1}} (\det(B_S[i]))(\det(B_S[i])) \\ &= \tau(G),\end{aligned}$$

where the second equation follows by the Cauchy-Binet formula, and the third by Lemma 2. □

We can now turn to the proof of the lemma.

Proof of Lemma 2:

Assume without loss of generality that the edges of $B_S[i]$ are "directed" arbitrarily; that is, we can change the column corresponding to (i,j) from $e_i - e_j$ to $e_j - e_i$, since this only flips the sign of the determinant.

If $S \subseteq E$ with $|S| = n-1$, and S is not a spanning tree, then it must contain a cycle. We can then direct the edges around this cycle. If we then sum the columns of $B_S[i]$ corresponding to the cycle, we obtain the 0 vector, which implies that the columns of $B_S[i]$ are linearly dependent, and thus $\det(B_S[i]) = 0$.

Now we suppose that S is a spanning tree and continue via induction on n.

Base case If $n = 2$, then

$$B_S = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

so $B_S[i] = \pm 1$ and thus $\det(B_S[i]) = 1$.

Inductive case: Suppose the lemma statement is true for graphs of size $n-1$. Let $j \neq i$ be a leaf of the tree, and take (k,j) to be an edge incident to j. We exchange rows and columns so that (k,j) is last column, and j is last row; this may flip sign of determinant, but that doesn't matter. Then,

$$B_S[i] = \begin{array}{c} \begin{array}{cccc} & & & (k,j) \end{array} \\ \left[\begin{array}{ccc|c} & & & 0 \\ & & & 0 \\ & & & 1 \\ & & & 0 \\ \hline 0 & \dots & 0 & -1 \end{array}\right] \end{array}$$

Thus if we expand the determinant along the last row we get

$$|\det(B_S[i])| = |\det(B_{S-\{(k,j)\}}[i,j])| = 1.$$

The last equality follows by induction, since $S - \{(k,j)\}$ is a tree on the vertex set without j and we assumed that j is a leaf. □

2 Consequences of the Matrix-Tree Theorem

Once we have the matrix-tree theorem, there are a number of interesting consequences, which we explore in this section. Given a square matrix $A \in \mathbb{R}^{n\times n}$, let A_{ij} be matrix without row i column j (so $A[i] = A_{ii}$). Consider the cofactor matrix $C = (c_{ij})$, with entry $c_{ij} = (-1)^{i+j}\det(A_{ij})$ defining the i,j *cofactor* of A. Then we define the *adjugate* $\mathrm{adj}(A) = C^T$. Next, we will need the following fact.

Fact 3

$$A\,\mathrm{adj}(A) = \det(A)I.$$

By the matrix-tree theorem, the (i,i) cofactor of L_G is equal to $\tau(G)$. But we can say something even stronger.

Theorem 3 *Every cofactor of L_G is $\tau(G)$, so that*

$$\mathrm{adj}(L_G) = \tau(G)J.$$

Proof:

If G is not connected, then $\tau(G) = 0$ and $\lambda_2(L_G) = 0 = \lambda_1(L_G)$, so the rank of L_G is at most $n-2$. Then $\det((L_G)_{ij}) = 0$, which implies that $\mathrm{adj}(L_G) = 0$, as desired.

If G is connected, the fact above gives that $L_G\,\mathrm{adj}(L_G) = 0$ (i.e. the zero matrix) since $\det(L_G) = 0$. Because G is connected, multiples of e are the only eigenvectors of L_G with eigenvalue 0, and every column of $\mathrm{adj}(L_G)$ must therefore be some multiple of e. But we know that for the i^{th} column of $\mathrm{adj}(L_G)$, its i^{th} entry is $\tau(G)$, so the column itself must be $\tau(G)e$, and the lemma statement follows. □

We conclude with one more theorem.

Theorem 4 *Let* $0 = \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n$ *be the eigenvalues of* L_G*. Then*

$$\tau(G) = \frac{1}{n}\prod_{i=2}^{n}\lambda_i.$$

Proof:

If G is not connected, then $\lambda_2 = 0$ and $\tau(G) = 0$, so the theorem holds. Otherwise, we will look at linear term of the characteristic polynomial in two different ways. From one perspective, the characteristic polynomial is

$$(\lambda - \lambda_1)(\lambda - \lambda_2)...(\lambda - \lambda_n) = \lambda(\lambda - \lambda_2)(\lambda - \lambda_3)...(\lambda - \lambda_n),$$

so the linear term is

$$(-1)^{n-1}\prod_{i=2}^{n}\lambda_i.$$

On the other hand, we know that the characteristic polynomial is defined as $\det(\lambda I - L_G)$. We'll use that for A, B of the same dimensions, $\det(A + B) = \sum_{S\subseteq[n]} \det A_S$, where A_S is the matrix A with the rows indexed by S replaced by the corresponding rows of B. For $A = \lambda I$, $B = -L_G$, the linear term in λ is given by

$$\sum_{S\subseteq[n],|S|=n-1} \det(\lambda I)_S = (-1)^{n-1}\sum_{i=1}^{n}\det(L_G[i]) = (-1)^{n-1}\cdot n\cdot\tau(G).$$

Therefore, $\tau(G) = \frac{1}{n}\prod_{i=1}^{n}\lambda_i$.

□