ORIE 6334 Bridging Continuous and Discrete Optimization October 16, 2019

Lecture 12

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In this lecture, we will introduce planar graphs, and investigate the connections between this property and the spectrum of matrices associated with the graph.

1 Introduction to Planar Graphs

We will begin with an informal definition (a standard graph theory textbook will have a more rigorous definition).

Definition 1 (Planar) *A graph $G = (V, E)$ is planar if*

- *for each vertex $i \in V$ there exists a point $x_i \in \mathbb{R}^2$*
- *for each edge $(i, j) \in E$ there exists a curve between x_i and x_j that intersects no other curve*

Conceptually, observe that if a graph is planar it means that it can be drawn in the plane without any edges intersecting. We call this collection of points and curves a *planar embedding* of G. The term *plane graph* refers to the graph G and a planar embedding. The plane graph divides the $\mathbb{R}^2$ plane into regions called the *faces* of the graph. This includes the *external face*, which is the face formed by the outermost curves.

A planar graph G is *maximal* if adding any edge e to G makes $G+e$ non-planar. For any maximal planar graph, every face in a planar embedding must be a triangle, since otherwise we could add an edge. A graph H is a *minor* of G if we can obtain H from G by some sequence of deleting and/or contracting edges.

Recall that K_5 is the complete graph with five vertices and $K_{3,3}$ is the complete bipartite graph with 3 vertices on each side. Additionally, recall that 3-vertex-connected means that up to any two vertices can be removed from a graph and it is still connected. We will now state a couple of theorems about planar graphs without proof.

Theorem 1 (Kuratowski 1930, Wagner 1937) *A graph is planar if and only if it does not have K_5 or $K_{3,3}$ as a minor.*

Theorem 2 *Any maximal planar graph is 3-vertex-connected.*

2 Generalized Laplacian Matrix

We now switch gears and introduce a generalization of the Laplacian matrix concept that we have previously used. The formal definition, given below, is very similar to the original definition; however, there is no condition on the diagonal elements of the matrix.

[0]This lecture is derived from Godsil and Royle, Sections 1.8, 13.9-13.11; and Van der Holst 1995 (http://oai.cwi.nl/oai/asset/2232/2232A.pdf).

Definition 2 (Generalized Laplacian) *A generalized Laplacian of graph G is a symmetric matrix $M = (m_{ij}) \in \mathbb{R}^{n\times n}$ such that*

$$m_{ij} < 0 \text{ if } (i,j) \in E,$$
$$m_{ij} = 0 \text{ if } (i,j) \notin E \text{ and } i \neq j.$$

By following proofs from earlier in the course (for Laplacian matrices), we can assume that if G is connected then the eigenvalues λ_i are such that $\lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_n$ and the eigenvectors x_i are such that $x_1 > 0$.

Since the generalized Laplacian has no condition on the diagonal, then $M - \lambda I$ is a generalized Laplacian if M is. Thus, we can apply a linear shift to all of the eigenvalues of a generalized Laplacian matrix and still have a generalized Laplacian. So, we will assume that λ_1 is the unique negative eigenvalue and that $\lambda_2 = 0$. We now introduce the concepts of kernel and co-rank.

Definition 3 (Kernel & Co-Rank) *The kernel of M is*

$$\ker(M) = \{x \in \mathbb{R}^n : Mx = 0\}.$$

The co-rank of M is $\dim(\ker(M))$.

For G connected and generalized Laplacian M such that $\lambda_2(M) = 0$, the co-rank is the multiplicity of 0 as an eigenvalue.

Definition 4 (Colin de Verière invariant) *The* Colin de Verdière invariant $\mu(G)$ *is the largest corank of a generalized Laplacian such that:*

1. *M has exactly 1 negative eigenvalue;*
2. *There is no $X = (x_{ij}) \in \mathbb{R}^{n\times n}$ such that $X \neq 0$, $MX = 0$, $x_{ii} = 0$ for all $i \in V$, and $x_{ij} = 0$ if $m_{ij} \neq 0$. [Strong Arnold Property]*

3 Planarity via Generalized Laplacians

We will now begin to connect planar graphs with generalized Laplacians.

Theorem 3 (Colin de Verdière 1990) *The following hold:*

- *$\mu(G) \leq 1$ iff G is a collection of paths.*
- *$\mu(G) \leq 2$ iff G is outerplanar (planar and all vertices are on the external face).*
- *$\mu(G) \leq 3$ iff G is planar.*

The challenging direction of the final relationship is to show if G is not planar then $\mu(G) > 3$. The proof makes use of the fact that $\mu(K_{3,3}) = \mu(K_5) = 4$ and the following theorem.

Theorem 4 (Colin de Verdière 1990) *If H is a minor of G then $\mu(H) \leq \mu(G)$.*

The easier direction of the proof is if G is planar then $\mu(G) \leq 3$. We will now prove this direction via a proof given by Van der Holst in 1995.

Let the support of x be denoted by $\mathrm{supp}(x) = \{i \in V : x(i) \neq 0\}$, $\mathrm{supp}^+(x) = \{i \in V : x(i) > 0\}$, and $\mathrm{supp}^-(x) = \{i \in V : x(i) < 0\}$. We start by stating and proving a series of lemmas.

Lemma 5 *Suppose $x \in \ker(M)$ where M is a generalized Laplacian matrix. If $i \notin \mathrm{supp}(x)$, then either all of the neighbors of i are not in $\mathrm{supp}(x)$ or i has neighbors in both $\mathrm{supp}^+(x)$ and $\mathrm{supp}^-(x)$.*

Proof: If $Mx = 0$ then $(Mx)(i) = 0$ for each i. Then,

$$0 = (Mx)(i) = \sum_{j:(i,j)\in E} m_{ij}x(j) + m_{ii}x(i) = \sum_{j:(i,j)\in E} m_{ij}x(j)$$

since $i \notin \mathrm{supp}(x)$. Since $m_{ij} < 0$ for all j such that $(i,j) \in E$, either all $x(j) = 0$ or some are positive and others are negative. □

Lemma 6 *For $x \in \ker(M)$, where M is a generalized Laplacian, $x \neq 0$, G connected, then $\mathrm{supp}^+(x) \neq \emptyset$ and $\mathrm{supp}^-(x) \neq \emptyset$.*

Proof: If $x \in \ker(M)$, then it is in the span of the eigenvectors that have eigenvalue 0. These eigenvectors are orthogonal to the one eigenvector x_1 of negative eigenvalue, and we can assume that $x_1 > 0$ (as discussed in Section 2). Thus $x^T x_1 = 0$, and since $x \neq 0$ and $x_1 > 0$, x has both positive and negative entries. □

For the next lemma, we need to define another term.

Definition 5 (Minimal Support) *A vector x is said to have minimal support if $x \neq 0$ and for every $y \neq 0$ and $y \in \ker(M)$ with $\mathrm{supp}(y) \subseteq \mathrm{supp}(x)$ implies that $\mathrm{supp}(x) = \mathrm{supp}(y)$.*

Additionally, let's introduce some notation. Let $M[I, J]$ be the submatrix with rows from the index set I and columns from index set J.

Lemma 7 *Let G be a connected graph, M be a generalized Laplacian with only one negative eigenvalue. Let $x \in \ker(M)$ have minimal support. Then the graph induced by $\mathrm{supp}^+(x)$ (or by $\mathrm{supp}^-(x)$) is connected.*

Proof: We will give a proof by contradiction. Suppose that the graph induced by the positive support of x is not connected. Let I and J be a partition of $\mathrm{supp}^+(x)$ such that I and J are not connected. We will now show that we can find $y \in \ker(M)$ and $y \neq 0$ but $\mathrm{supp}(y) \subset \mathrm{supp}(x)$.

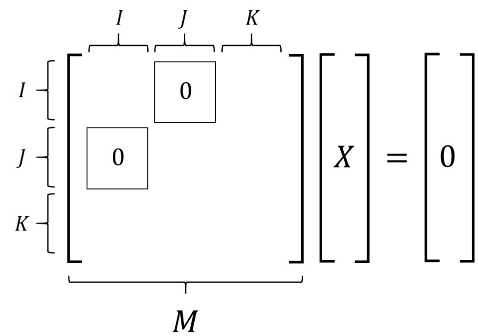

Figure 1: Visualization of the $Mx = 0$ equation.

Let $K = \text{supp}^-(x)$. Since $Mx = 0$ and $M[I, J] = M[J, I] = 0$ (because I and J are not connected), and x is zero on the indices outside I, J, K, we have

$$M[I, I]x[I] + M[I, K]x[K] = 0 \quad (1)$$
$$M[J, J]x[J] + M[J, K]x[K] = 0 \quad (2)$$

Let x_1 be the eigenvector corresponding to the single negative eigenvalue. We know that $x_1 > 0$. Now, define α as

$$\alpha = \frac{x_1[I]^T x[I]}{x_1[J]^T x[J]}$$

and note that $\alpha > 0$ because all the terms in it are positive. Now, define y such that

$$y = \begin{cases} x(i) & \forall i \in I \\ -\alpha x(i) & \forall i \in J \\ 0 & \text{else} \end{cases}$$

We claim that $\text{supp}(y) \subset \text{supp}(x)$; this follows because we know that $K = \text{supp}^-(x) \neq \emptyset$. Further note that

$$x_1^T y = x_1[I]^T x[I] - \alpha x_1[J]^T x[J] = 0,$$

so that y is orthogonal to x_1. We also have that

$$\begin{aligned} y^T M y &= y[I]^T M[I, I] y[I] + y[J]^T M[J, J] y[J] \\ &= x[I]^T M[I, I] x[I] + \alpha^2 x[J]^T M[J, J] x[J] \\ &= -x[I]^T M[I, K] x[K] - \alpha^2 x[J]^T M[J, K] x[K] \leq 0, \end{aligned}$$

where the last equality uses (1) and (2), and the final inequality follows because $x[I], x[J] > 0$, $x[K] < 0$, $M[I, K], M[J, K] \leq 0$ and $\alpha > 0$.

So we have that $x_1^T y = 0$ and $y^T M y \leq 0$. Because y is orthogonal to x_1 (the one eigenvector of negative eigenvalue), it must also be the case that $y^T M y \geq 0$. Thus $My = 0$, which means that $y \in \ker(M)$, implying that x does not have minimal support. This is the desired contradiction. □

We can now prove the following.

Theorem 8 *If G is planar and 3-vertex-connected, then $\mu(G) \leq 3$.*

Proof: We will prove this by contradiction. Let's suppose G is planar but $\mu(G) > 3$. We show that we can find a $K_{3,3}$ minor in G.

Choose some plane embedding of G and choose a face. Let u, v, w be on the face. Because $\dim(\ker(M)) \geq 4$, there exists $x \in \ker(M)$ such that $u, v, w \notin \text{supp}(x)$. Assume that x has minimal support.

Now let's add another point s into the face, and add edges from s to u, v, w. Pick $p \in \text{supp}(x)$. Since G is 3-vertex-connected, there exist vertex disjoint paths from p to u, v, w. Let a, b, c be the first vertices on these paths such that they are not in the support of x and all subsequent vertices are not in the support of x (see Figure 2). We know that $a, b, c \notin \text{supp}(x)$ have neighbors in the support of x, and thus by Lemma 5 we know that they must have neighbors both in $\text{supp}^+(x)$ and $\text{supp}^-(x)$ (see Figure 3).

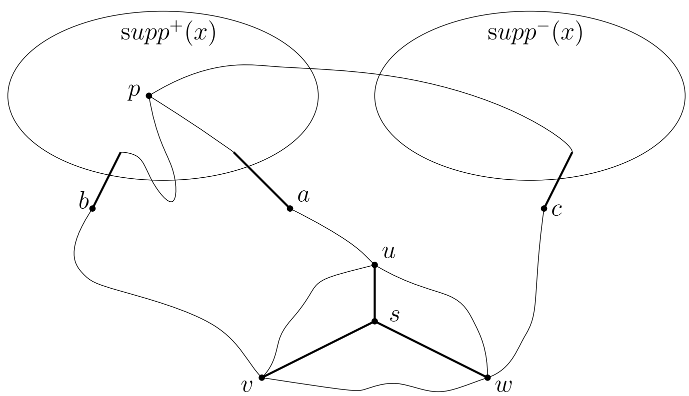

Figure 2: Vertex disjoint paths from p to u, v, w on a single face, with s added to the face.

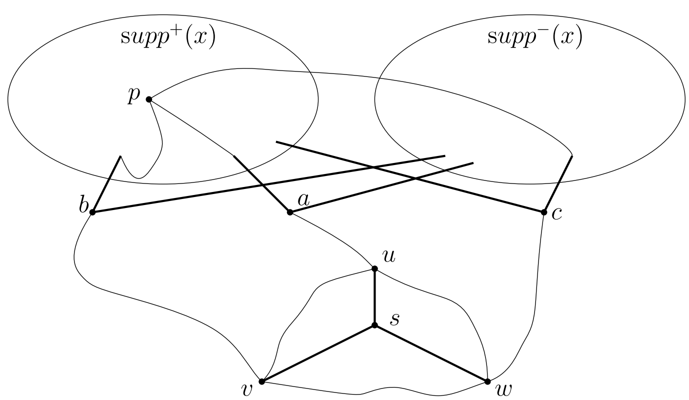

Figure 3: a, b, c must have neighbors in both $\text{supp}^+(x)$ and $\text{supp}^-(x)$.

Now, let's contract $\text{supp}^+(x)$ and $\text{supp}^-(x)$ to single vertices s^+ and s^-, respectively (see Figure 4); we can do this since $\text{supp}^+(x)$ and $\text{supp}^-(x)$ are each connected. Now, let's also then contract the $a-u$, $b-v$, and $c-w$ paths. This leaves us with $\{s, s^+, s^-\}$ and $\{a, b, c\}$ forming the $K_{3,3}$ complete graph, and so we have completed the proof. □

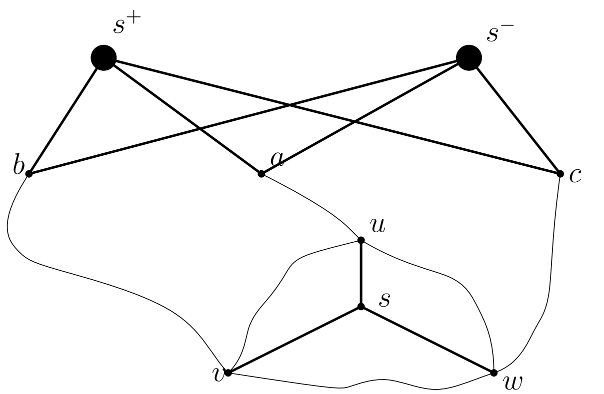

Figure 4: Contracting $\mathrm{supp}^+(x)$ and $\mathrm{supp}^-(x)$ to s^+ and s^- respectively.

With a little twist via the closing corollary, we can complete the proof of that direction of any G.

Corollary 9 *Since $\mu(G)$ only increases when adding edges, any maximal planar graph is 3-vertex-connected. Thus, we can make any G maximally planar, so $\mu(G) \leq 3$ for any planar G.*