

Lecture 1

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The general theme of this class will be to link ideas that at first sight seem completely unrelated. Today we begin with a classical example involving graphs and eigenvalues.

1 Eigenvectors, Eigenvalues, and Graph Theory

Let us recall a couple of definitions. Consider a symmetric matrix $A \in \mathbf{R}^{n \times n}$. We say that $x \in \mathbf{R}^n$ is an *eigenvector* and λ is an *eigenvalue* if $Ax = \lambda x$. Eigenvectors and eigenvalues have applications in differential equations, mechanics, frequency analysis, and many others.

An *undirected graph* G is represented as a tuple (V, E) consisting of a set of vertices V and a set of edges E . We are interested in paths, flows, cuts, colorings, cliques, spanning trees, among others.

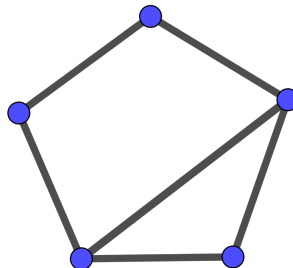
During part of this semester, we will ask what graphs and eigenvalues have to do with each other.

2 An Introductory Example

The diameter D of a graph is minimum length you would have to be able to travel to guarantee that you could go from any node in the graph to any other node. Formally,

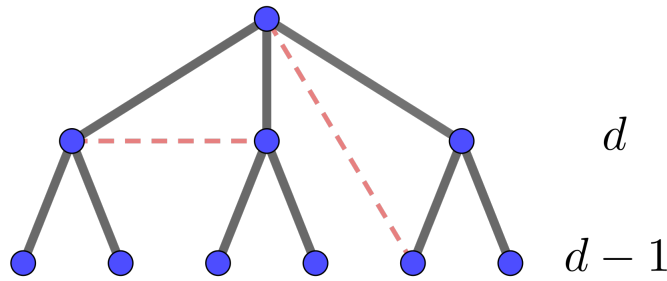
$$D = \max_{i,j \in V} \text{“length of shortest path between } i \text{ and } j\text{”}.$$

A graph is said to be d -regular if all nodes are of degree d , where degree is defined as the number of edges incident on each vertex. The below graph has diameter 2 but is not d -regular since some nodes are of degree 2 and some are of degree 3.



For our introductory example¹, we will consider d -regular graphs of diameter 2 with as many nodes as possible. By starting at any node i , the graph could look like

¹This material is taken from the article A.J. Hoffman, R.R. Singleton, “On Moore Graphs with Diameters 2 and 3,” *IBM Journal*, pp. 497–504, November 1960.



In this graph, note that there are no connections between adjacent nodes in the first layer since we want to maximize the number of nodes in the graph. In fact, there cannot be triangles nor squares in this type of graph. Also, the connections between the leaf nodes are omitted from the diagram. Based on the diagram, such a graph would have $n = 1 + d + d(d - 1) = d^2 + 1$ nodes.

Let $A = (a_{ij})$ be the adjacency matrix of G , defined as

$$a_{ij} = \begin{cases} 1 & \text{if } (i, j) \in E \\ 0 & \text{otherwise.} \end{cases}$$

If $B = A^2$, then

$$b_{ij} = \sum_k a_{ik}a_{kj} = \text{number of walks of 2 steps in graph } G \text{ from } i \text{ to } j.$$

Since the graph is d -regular, we have that $b_{ii} = d$. This follows because starting at node i we can reach d vertices in one step and then immediately return. Additionally, it is not hard to see that starting at i we reach every other vertex in G in exactly 0 steps, exactly 1 step, or exactly 2 steps (exclusive or). Thus, the following holds

$$I + A + A^2 - dI = J,$$

where I is the identity matrix and J is the matrix of all ones.

We'll need the following facts from linear algebra.

Fact 1 For $A \in \mathbf{R}^{n \times n}$ symmetric, the following are true:

- All of the eigenvalues of A are real.
- There exist eigenvalues $\lambda_1, \dots, \lambda_n$ (called the spectrum) and eigenvectors x_1, \dots, x_n such that $\langle x_i, x_j \rangle = x_i^T x_j = 0$ for $i \neq j$.
- The trace $\text{tr}(A) = \sum_{i=1}^n a_{ii} = \sum_{i=1}^n \lambda_i$.

Existence. We've discussed the notion of a d -regular graph with diameter two, *but does such a graph exist?* To answer this question we will use linear algebra.

Let us start by noticing that $Ae = de$, where $e = (1, \dots, 1)^T$ is the vector of all-ones (which follows since the graph is d -regular). Thus, e is an eigenvector of A and d is the eigenvalue. Observe that

$$A^2e = A(Ae) = A(de) = d(Ae) = d^2e.$$

Thus,

$$\begin{aligned}(I + A + A^2 - dI)e &= Je \\ e + de + d^2e - de &= ne\end{aligned}$$

So $n = d^2 + 1$, though we already knew that.

Now let v be any other eigenvector of A orthogonal to e . Then $v^T e = 0$, and thus $Jv = 0$. We have that $Av = \lambda v$ for some eigenvalue λ . Also, $A^2v = A(Av) = A(\lambda v) = \lambda^2 v$. Thus

$$\begin{aligned}(I + A + A^2 - dI)v &= Jv \\ v + \lambda v + \lambda^2 v - \lambda v &= 0 \\ \implies 1 + \lambda + \lambda^2 - d &= 0\end{aligned}$$

So for all eigenvalues not corresponding to e , we have $\lambda = \frac{-1 \pm \sqrt{4d-3}}{2}$.

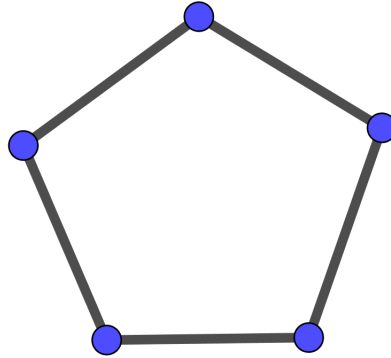
Given what we now know about the eigenvalues, what can we tell? We can invoke the trace! Notice that $\text{tr}(A) = 0$: $a_{ii} = 0$ for all i since there are no self-loops in the graph. Now we consider two possible cases.

Case 1. If $\sqrt{4d-3}$ is irrational, then in order for the trace to sum to zero, the eigenvalues $\frac{-1 + \sqrt{4d-3}}{2}$ and $\frac{-1 - \sqrt{4d-3}}{2}$ must each have multiplicity $\frac{n-1}{2}$. Plugging this in gives

$$\begin{aligned}\text{tr}(A) = 0 &= d + \frac{n-1}{2} \left(\frac{-1 + \sqrt{4d-3}}{2} + \frac{-1 - \sqrt{4d-3}}{2} \right) \\ &= d - \frac{n-1}{2} \\ &= d - \frac{d^2}{2} \\ \implies d &= 0 \text{ or } d = 2.\end{aligned}$$

So the only possible graphs would be:

- (a) Assume $d = 0$, then the graph is a single node, which does not have diameter two.
- (b) Assume $d = 2$, in this case $n = 5$. This gives the 5 cycle; the 5-cycle is a 2-regular graph of diameter two.



Case 2. If $\sqrt{4d-3}$ is rational, then let $s^2 = 4d-3$. Let m be the multiplicity of the eigenvalue $\frac{-1+s}{2}$. Then

$$\begin{aligned} \text{tr}(A) &= d + m \left(\frac{-1+s}{2} \right) + (n-1-m) \left(\frac{-1-s}{2} \right) \\ &= 0 \end{aligned}$$

Using the fact that $d = \frac{1}{4}(s^2 + 3)$, we get $n-1 = d^2 = \frac{1}{16}(s^4 + 6s^2 + 9)$. So continuing

$$0 = \frac{1}{4}(s^2 + 3) + m \left(\frac{-1+s}{2} \right) + \left(\frac{1}{16}(s^4 + 6s^2 + 9) - m \right) \left(\frac{-1-s}{2} \right)$$

After simplifying the algebra, we find that

$$-s^5 - s^4 - 6s^3 + 2s^2 + (32m - 9)s + 15 = 0.$$

By the rational root theorem, we know that any solution to this polynomial must be a factor of 15. Thus, we can enumerate all possible roots.

Hence we have the following possibilities:

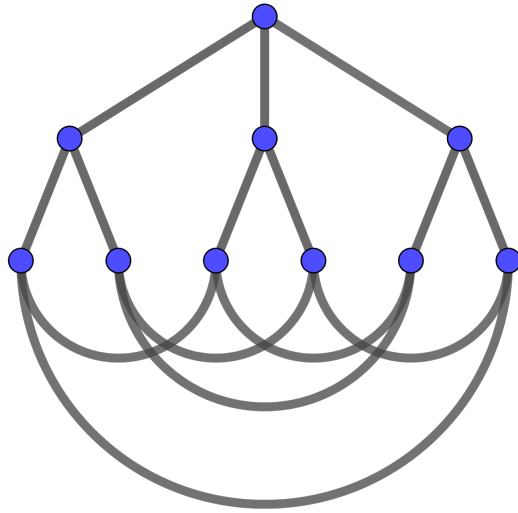
- (a) Assume $s = 1, d = 1, n = 2$:

A 1-regular graph on 2 nodes is a single edge, but its diameter is not 2.

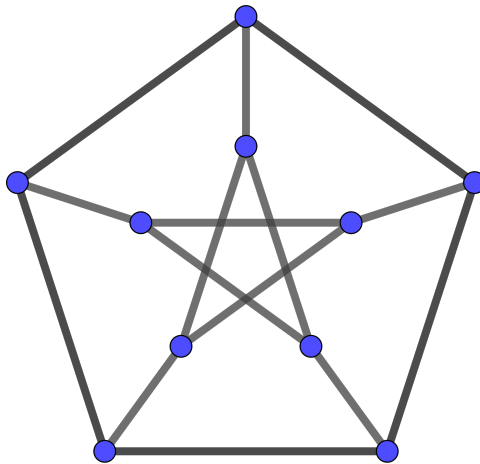


- (b) Assume $s = 3, d = 3, n = 10$:

We can represent this graph in two ways:



This first representation shows what the graph would look like using the same type of diagram we used earlier.

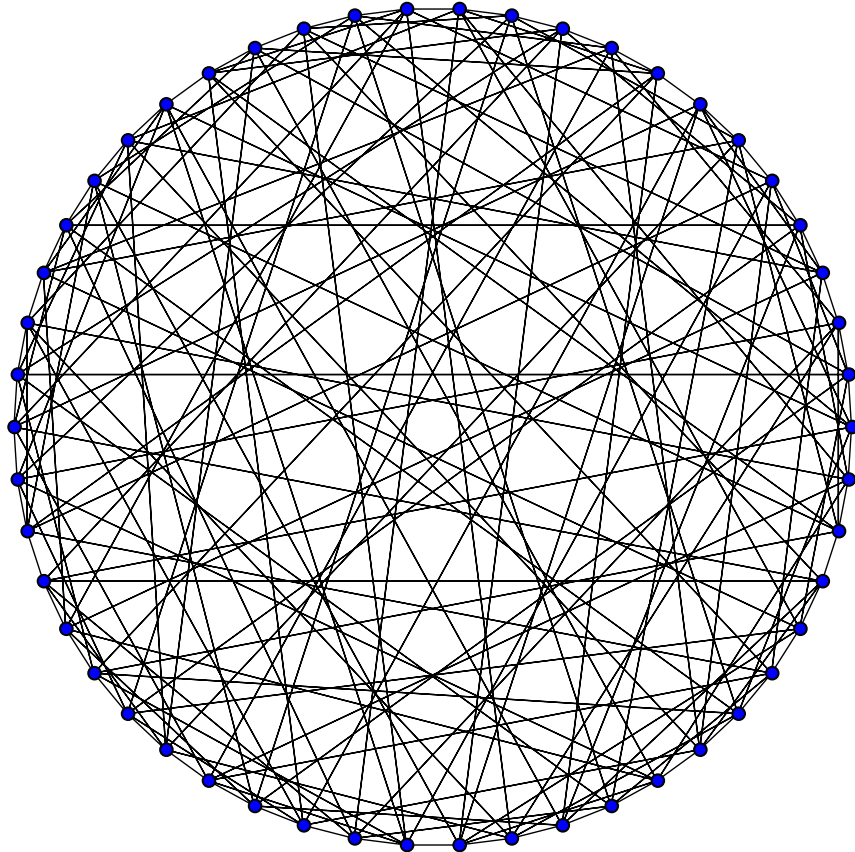


This second representation is often called the Petersen representation, and the graph is called the *Petersen graph*. Petersen found it in the course of trying to find the smallest cubic (that is, 3-regular) bridgeless graph that could not be 3-edge-colored. A bridgeless graph is one such that the graph is still connected after removing any edge. A 3-edge-colorable graph is one in which we can color every edge with one of three colors such that at each vertex, all incident edges have different colors. The Petersen graph is also the smallest cubic bridgeless graph that does not have a Hamiltonian cycle. Knuth has called the Petersen graph:

“A remarkable configuration that serves as a counterexample to many optimistic predictions about what might be true for graphs in general.”

(c) Assume $s = 5, d = 7, n = 50$:

This graph is known to exist and is called the Hoffman-Singleton graph (Hoffman, Singleton 1960).



(Image source: Wikipedia)

(d) Assume $s = 15, d = 57, n = 3250$:

Does this graph exist? We don't know. This is a good research question!

This example is to give you a small taste of how eigenvectors can be useful in graph theory. By looking at the spectrum of d -regular graphs of diameter 2 with as many nodes as possible, we were able to come up with very strong restrictions on the possible values of d .