ORIE 633 Network Flows	October 16, 2007
Lecture 12	
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1 Minimum-cost circulations

Recall the minimum-cost circulation problem, introduced in the previous lecture:

Minimum-cost circulation problem

- **Input:**
  - A directed graph $G = (V, A)$.
  - Integer costs $c_{ij} \in \mathbb{Z}$, $\forall (i,j) \in A$.
  - Integer capacities $u_{ij} \geq 0$, $\forall (i,j) \in A$.
  - Integer demands $0 \leq l_{ij} \leq u_{ij}$, $\forall (i,j) \in A$.
- **Goal:** Find a minimum-cost circulation.

The goal is to find a flow $f : A \to \mathbb{R}^{\geq 0}$ that minimizes $\sum_{(i,j)\in A} c_{ij} f_{ij}$ such that

$$l_{ij} \leq f_{ij} \leq u_{ij}, \qquad \forall (i,j) \in A$$
$$\sum_{k:(i,k)\in A} f_{ik} - \sum_{k:(k,i)\in A} f_{ki} = 0, \qquad \forall i \in V$$

In the previous lecture, we defined a notation change for circulations similar to the one we defined for s-t flows.

Definition 1 *A* circulation f *satisfies the following:*

1. $f_{ij} \leq u_{ij}\ \forall\ (i,j) \in A$
2. $f_{ij} = -f_{ji},\ \forall\ (i,j) \in A$
3. $\sum_{k:(k,i)\in A} f_{ki} = 0$

In the new definition, flow in the original arc f_{ij} satisfies the constraints $f_{ij} \leq u_{ij}$, and each unit of flow incurs cost c_{ij}. Flow on the reverse arc f_{ji} satisfies $f_{ji} \leq u_{ji} = -l_{ij}$ and incurs cost $c_{ji} = -c_{ij}$ per unit of flow. The total cost for the two edges with flow f is $c_{ji}f_{ji} + c_{ij}f_{ij} = 2c_{ij}f_{ij}$. Hence optimizing the total cost for this new graph is the same as optimizing the total cost for the original graph.

Given a flow f on G, last time we defined the *residual graph* to be $G_f = (V, A_f)$ where the new arc set

$$A_f := \{(i,j) \in A : f_{ij} < u_{ij}\}.$$

Note that we are using the new notation here. Impose the upper bound $u^f_{ij} = u_{ij} - f_{ij}$ for arc $(i,j) \in A_f$. Then clearly $u^f_{ij} > 0$ for all $(i,j) \in A_f$.

We also defined node potentials.

Definition 2 *A* potential *is a function* $p : V \to \mathbb{R}$.

Definition 3 *Given a potential* p*, define the* reduced cost $c^p_{ij} := c_{ij} + p_i - p_j$. *Then* $c^p_{ji} = c_{ji} + p_j - p_i = -(c_{ij} + p_i - p_j) = -c^p_{ij}$.

The node potentials play the role of dual variables.

Definition 4 *The cost of a cycle* Γ *is* $c(\Gamma) = \sum_{(i,j)\in\Gamma} c_{ij}$.

We proved the following theorem (optimality conditions) in the last lecture:

Theorem 1 *The following are equivalent:*

1. f *is a minimal cost circulation,*
2. *There are no negative cost cycles in* G_f*, and,*
3. *There exists a potential* p *such that* $c^p_{ij} \geq 0$ *for all* $(i,j) \in A_f$.

2 A cycle-cancelling algorithm

The theorem above leads to a natural algorithm for computing a min-cost circulation:

Cycle-Cancelling Algorithm (Klein '67)

Let f be a feasible circulation.
While A_f contains a negative cycle Γ
    Cancel Γ, update f.

The correctness of the algorithm follows immediately from the above theorem. If costs and capacities are both integral, then there exists an optimal flow f such that f_{ij} integer for all $(i,j) \in A$. Suppose

$$U = \max_{(i,j)\in A} u_{ij} \qquad C = \max_{(i,j)\in A} |c_{ij}|.$$

Then any feasible circulation costs at most mCU and at least $-mCU$. Since a cycle cancellation improves the cost of a circulation by at least 1, at most $O(mCU)$ cancellations are needed in order to find an optimal circulation.

We need two more things to conclude that the above algorithm is pseudo-polynomial:

1. We need to be able to find an initial circulation: For this, recall from Problem Set 1 that this can be done in one max flow calculation.
2. We need to be able to check the existence of a negative-cost cycle: For this, recall from Problem Set 2 that the this can be done via the Bellman-Ford algorithm in $O(mn)$ time. So we have a pseudo-polynomial time algorithm that runs in $O(m^2nCU)$ time.

3 Minimum mean-cost cycle cancelling

As with the augmenting path algorithm for the maximum flow problem, we can obtain a polynomial-time algorithm by a better choice of which cycle to cancel at each iteration. Consider the following.

Definition 5 *Let the* mean cost *of a cycle* Γ *be* $\frac{c(\Gamma)}{|\Gamma|}$ *where* $c(\Gamma)$ *is the cost of the cycle and* $|\Gamma|$ *is the number of arcs in* Γ*.*

Definition 6 *Given a circulation* f*, let* $\mu(f)$ *be the cost of the minimum mean-cost cycle in* G_f*:*

$$\mu(f) = \min_{\text{cycles } \Gamma \subseteq A_f} \frac{c(\Gamma)}{|\Gamma|}$$

It turns out we can get a polynomial-time algorithm by cancelling the *minimum mean-cost cycle.* We can now give the following algorithm:

Minimum mean-cost cycle cancelling algorithm (Goldberg-Tarjan '89)

> Let f be any circulation
> While $\mu(f) < 0$
>   Cancel min-mean cycle Γ, update f

Observe that the condition $\mu(f) < 0$ is equivalent to having a negative-cost cycle in A_f. To have a polynomial-time algorithm, we need to be able to find the minimum mean-cost cycle in polynomial-time. In Problem Set 3, we will show that one can compute $\mu(f)$ and find the corresponding cycle in $O(mn)$ time.

To begin our analysis, we need to introduce a few terms.

Definition 7 *A circulation* f *is* ϵ*-optimal if there exist potentials* p *s.t.* $c^p_{ij} \geq -\epsilon$ *for all* $(i,j) \in A_f$*.*

Clearly f is 0-optimal if and only if f is a min-cost circulation by the third equivalence in Theorem 1. For any circulation, f is $C-$optimal, since if we assign $p_i = 0$ for all $i \in V$, $c^p_{ij} \geq -C$ for all $(i,j) \in A_f$.

Definition 8 *Define* $\epsilon(f)$ *to be the minimum* ϵ *such that* f *is* ϵ*-optimal.*

Interestingly, the two values of $\epsilon(f)$ and $\mu(f)$ are closely related.

Theorem 2 *For a circulation* f*,* $\mu(f) = -\epsilon(f)$*.*

Proof: We first show that $\mu(f) \geq -\epsilon(f)$. Since $\exists\, p$ s.t. $c^p_{ij} \geq -\epsilon(f)$ for all $(i,j) \in A_f$, by summing over all arcs in any cycle Γ we obtain that $c^p(\Gamma) \geq -\epsilon(f)|\Gamma|$. Thus

$$\mu(f) = \frac{c(\Gamma)}{|\Gamma|} = \frac{c^p(\Gamma)}{|\Gamma|} \geq -\epsilon(f).$$

for a minimum mean-cost cycle Γ.

We now show that $\mu(f) \leq -\epsilon(f)$. Set $\overline{c}_{ij} = c_{ij} - \mu(f)$. Then for any cycle Γ in A_f:

$$\overline{c}(\Gamma) = c(\Gamma) - |\Gamma|\mu(f) \geq c(\Gamma) - |\Gamma|\frac{c(\Gamma)}{|\Gamma|} = 0.$$

We introduce a source vertex s, connected to all vertices i with arcs of cost $\overline{c}_{si} = 0$, and define the potential p_i of node i to be the length of shortest path from s to i using costs $\overline{c}_{ij}$. Note that this notion is well-defined, since by the previous argument, there are no negative-cost cycles with respect to costs $\overline{c}_{ij}$. By the definition of shortest path, for all $(i,j) \in A_f$, $p_j \leq p_i + \overline{c}_{ij} = p_i + c_{ij} - \mu(f)$ which implies $c^p_{ij} = c_{ij} + p_i - p_j \geq \mu(f)$ for all $(i,j) \in A_f$. This means f is $-\mu(f)$-optimal which implies that $\epsilon(f) \leq -\mu(f)$. □

Given circulation f, let $f^{(k)}$ denote the circulation we get after k iterations of cancelling minimum mean-cost cycles in f. The following theorems, which we will prove later, will show that the Goldberg-Tarjan algorithm runs in polynomial time.

Theorem 3 *$\epsilon(f^{(1)}) \leq \epsilon(f)$.*

Theorem 4 *$\epsilon(f^{(m)}) \leq (1 - 1/n)\epsilon(f)$.*

where m, n are the number of arcs and nodes in the graph, respectively.

We will also need the following.

Theorem 5 *When $\epsilon(f) < 1/n$ then circulation f is optimal.*

Proof: The fact that $\epsilon(f) < 1/n$ implies that there exist a potential p such that $c^p_{ij} > -1/n$ for all $(i,j) \in A_f$. Thus for all cycles $\Gamma \in A_f$, $c(\Gamma) = c^p(\Gamma) > -1$. By the integrality of costs, this gives $c(\Gamma) \geq 0$. □

We shall now prove using the previous three results that the Goldberg-Tarjan algorithm terminates in time bounded by a polynomial in the input size.

Theorem 6 *(Goldberg-Tarjan '89) The Goldberg-Tarjan minimum mean-cost cycle cancelling algorithm requires at most $O(mn\log(nC))$ iterations.*

Proof: Any initial circulation is C-optimal. After $k = mn\log(nC)$ iterations, we have that

$$\epsilon(f^{(k)}) \leq (1 - 1/n)^{n\log(nC)}C < e^{-\log(nC)}C = 1/n,$$

using the fact that $(1-1/n)^n < e^{-1}$. This proves the optimality of $f^{(k)}$ by Theorem 5. □

The running of the Goldberg-Tarjan algorithm is $O(m^2n^2\log(nC))$ time as min-mean cycle computations take $O(mn)$ time (See Problem Set 3 regarding the latter fact). Note that this algorithm is not strongly polynomial. A strongly polynomial algorithm will be presented in the next lecture. For now, we return and prove Theorem 3 and Theorem 4.

Proof of Theorem 3: We know there exist potentials p such that

$$c^p_{ij} \geq -\epsilon(f) \text{ for all } (i,j) \in A_f.$$

Also, $\mu(f) = -\epsilon(f)$. For the minimum-mean cost cycle Γ, since $\mu(f) = c^p(\Gamma)/|\Gamma|$, it follows that for all $(i,j) \in \Gamma$, $c^p_{ij} = -\epsilon(f)$. We now claim that $c^p_{ij} \geq -\epsilon(f)$ for all $(i,j) \in A_{f^{(1)}}$. We have $(i,j) \in A_{f^{(1)}}$ if either (i,j) was in A_f, or if $(j,i) \in \Gamma$. In the first case, $c^p_{ij} \geq -\epsilon(f)$. In the latter case, $c^p_{ij} = -c^p_{ji} = \epsilon(f) \geq 0$. In both cases, it follows that $f^{(1)}$ is $\epsilon(f)$-optimal, so the theorem statement follows. □

Proof of Theorem 4: We know there exists a potential p such that $c^p_{ij} \geq -\epsilon(f)$ for all $(i,j) \in A_f$. Suppose that in some iteration k we cancel cycle Γ such that $\exists (i,j) \in \Gamma$ with $c^p_{ij} \geq 0$. Then:

$$\begin{aligned} -\epsilon(f^{(k)}) = \mu(f^{(k)}) &= \frac{c^p(\Gamma)}{|\Gamma|} \\ &\geq \frac{|\Gamma|-1}{|\Gamma|}(-\epsilon(f)) \\ &\geq \left(1 - \frac{1}{n}\right)(-\epsilon(f)). \end{aligned}$$

Thus

$$\epsilon(f^{(k)}) \leq \left(1 - \frac{1}{n}\right)\epsilon(f).$$

How many consecutive iterations can there be such that $c^p_{ij} < 0$ for all (i,j) in the cancelled cycle Γ ? Cancelling the cycle removes one edge with $c^p_{ij} < 0$ from the residual graph and creates only edges with $c^p_{ij} \geq 0$. So we need no more than m iterations before we cancel such a cycle Γ. □