

Recitation 11

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Topic: Lagrangean Relaxation for Integer Program

We have seen a constraint-generation procedure to aid in solving the LP relaxation of the traveling salesman problem. This week we will consider another method for producing a bound on the optimal IP value.¹

In general, consider an integer program of the form:

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax \geq b \\ & Dx \geq d \\ & x \text{ integer} \end{aligned}$$

Assume all of the input data is integer-valued and suppose we can “quickly” optimize over the set

$$X = \{x : Dx \geq d, x \text{ integer}\}.$$

Now, we will relax the “hard” constraints $Ax \geq b$ by removing them and inserting a penalty for violations. Let p be a non-negative vector in \mathbf{R}^n , and consider the new problem:

$$\begin{aligned} \min \quad & cx + p(b - Ax) \\ \text{s.t.} \quad & x \in X. \end{aligned}$$

Let $Z(p)$ be the optimal objective value of this LP. Clearly, $Z(p)$ is no larger than the optimal IP value since the optimal IP solution is feasible for this problem and it satisfies $Ax_{IP}^* \geq b$, so $p(b - Ax_{IP}^*) \leq 0$.

The Lagrangean Dual

For any $p \geq 0$, we have $Z(p) \leq Z_{IP}$, giving us a bound on the optimal IP value. to get the best possible bound, consider the problem:

$$Z_D = \max_{p \geq 0} Z(p) = \max_{p \geq 0} \min_{x \in X} cx + p(b - Ax)$$

If $X = \{x^1, \dots, x^k\}$ is a finite set, then we can compute the value Z_D by the following LP:

$$\begin{aligned} \max \quad & q \\ \text{s.t.} \quad & q \leq cx^i + p(b - Ax^i) \quad i = 1, \dots, k \\ & p \geq 0 \end{aligned}$$

Note that when X is large, this is inefficient. However taking the dual of this LP, we get:

$$\begin{aligned} \min \quad & \sum_j y_j (cx^j) \\ \text{s.t.} \quad & \sum_j y_j (A_i x^j - b_i) \geq 0 \quad \forall i = 1, \dots, m \\ & \sum_j y_j = 1 \\ & y \geq 0 \end{aligned}$$

¹Based on previous notes of Maurice Cheung

If we rearrange the equations, and use the fact that $\sum_j y_j = 1$, we get an equivalent representation:

$$\begin{aligned} \min \quad & c \left(\sum_j y_j x^j \right) \\ \text{s.t. :} \quad & A \left(\sum_j y_j x^j \right) \geq b \\ & \sum_j y_j = 1 \\ & y \geq 0 \end{aligned}$$

Letting $\text{conv}(X)$ be the convex hull of X , note that $x \in \text{conv}(X)$ iff $x = \sum_j \alpha_j x^j$, $\sum_j \alpha_j = 1$, $\alpha_j \geq 0$, $x^j \in X$. Hence, this LP is exactly the same as optimizing over the convex hull of X . Hence, this can be written as:

$$\begin{aligned} \min \quad & cx \\ \text{s.t.} \quad & Ax \geq b \\ & x \in \text{conv}(X) \end{aligned}$$

Hence, Z_D can be computed by solving this LP. Note that an immediate corollary of this is that $Z_{LP} \leq Z_D$ since $\text{conv}(X) \subseteq \{x : Dx \geq d\}$. That means the bound provided by the Lagrangean dual is as least as strong as the LP-bound. Additionally, note that the analysis here can be extended to the case where X is not finite.

The Held-Karp Bound for Traveling Salesman Problem

Consider the TSP, that is, the problem of finding a minimum cost tour in a graph. In a previous recitation, we used the following characterization of tours: A subgraph forms a tour iff each vertex has degree 2 and each cut has at least 2 edges crossing it. This led to an LP relaxation that could be used to give a bound on the value of the optimal tour. Here, we will use a slightly different characterization. Number the nodes in the graph 1 through n for some arbitrary numbering.

Definition 1 A subgraph is a 1-tree if it is a spanning tree on nodes $\{2, \dots, n\}$ along with two edges incident to node 1.

Claim 1 A subgraph is a tour iff it is a 1-tree where each vertex $2, \dots, n$ has degree 2.

Thus, the following IP solves for the minimum cost tour, where V_{-1} is the set of vertices $\{2, \dots, n\}$, and $E(S)$ is the set of edges with both endpoints in S :

$$\begin{aligned} \min \quad & \sum_e c_e x_e \\ \text{s.t.:} \quad & \sum_{e \in E(V_{-1})} x_e = n - 2 \\ & \sum_{e \in E(S)} x_e \leq |S| - 1 \quad \forall S \subset V_{-1} \\ & \sum_{e \in \delta(\{1\})} x_e = 2 \\ & \sum_{e \in \delta(\{i\})} x_e = 2 \quad \forall i \in V_{-1} \\ & x \text{ integer} \end{aligned}$$

Let X be the set of vectors that satisfy all of these constraints except for the vertex constraints for $i \in V_{-1}$. These constraints say that there must be $n - 2$ edges in $E(V_{-1})$ and any subgraph on k nodes in V_{-1} can have at most $k - 1$ edges. Note that this implies that there are no cycles on that subgraph, therefore it must be a tree. Since we also require that there are two edges incident to node 1, the set X is exactly the set of 1-trees.

Thus, we can consider dualizing the vertex constraints for vertices in V_{-1} . The bound Z_D obtained this way is known as the Held-Karp lower bound. This bound is actually fairly easy to calculate. Since the set X is the set of 1-trees, we can calculate the min cost 1-tree for a particular vector p without solving an LP at all.

In this case, it turns out that the polytope defined by the non-dualized constraints has integer extreme points. Using the notation from the previous section, this says that $\{x : Dx \geq d\} = \text{conv}(X)$. Thus, for this formulation, $Z_{LP} = Z_D$. So the value of the Held-Karp lower bound is not in the strength of the bound, but in the efficient computation.