Problem Set 6

Due Date: October 10, 2014

1. (7 points) When I was a researcher at IBM, I once needed to solve several linear programs that had a very large number of constraints relative to the number of variables[1]. I was trying to solve $\min c^T x : Ax \leq b, x \geq 0$, for $A \in \Re^{m \times n}$, where m was exponential in n. To get this into standard form, add slack variables s for each constraint, so that the problem becomes $\min c^T x : Ax + s = b, x \geq 0, s \geq 0$. The running time for solving each LP was several hours. I wandered down the hall and asked John Forrest, the author of IBM's linear programming code (OSL), if there was anything I could do to speed up the solution time. He suggested a simple reformulation of my problem that dropped the running time to under 20 minutes. Try to explain a change such that the simplex method as we have described it (running on the primal in standard form) might run much more quickly in such a situation. Be sure to justify your answer; that is, you should explain why the reformulation would cause the simplex method to run more quickly. (Hint: think about the complexity of a pivot operation).

2. (Bertsimas and Tsitsiklis 3.6) Let x be a basic feasible solution with associated basis B. Prove the following:

   (a) (4 points) If the reduced cost of every nonbasic variable is positive, then x is the unique optimal solution.

   (b) (4 points) If x is the unique optimal solution and is nondegenerate, then the reduced cost of every nonbasic variable is positive.

3. Consider the linear program $\min(cx : x \geq 0, Ax = b)$. Let B denote an optimal basis. Assume that the problem is generic in that each vertex has a unique basis for which it is the corresponding basic solution.

   Assume now that you want to solve a *parametric* problem, i.e., a set of problems of the form $\min((c + \lambda d)x : x \geq 0, Ax = b)$, for each possible value of $\lambda \geq 0$. The basis B is a solution for the problem when $\lambda = 0$.

   (a) (7 points) Prove that the set of values of λ for which basis B is optimal forms an interval $[0, a_1]$. Explain how to compute a_1.

   (b) (10 points) Show that there is a finite set $a_0 = 0 \leq a_1 \leq \ldots \leq a_k$ and corresponding bases B_i for $i = 0, \ldots, k$ such that $B_0 = B$ and B_i (for $i = 0, \ldots, k$) is the optimal basis if and only if $\lambda \in [a_i, a_{i+1}]$, and B_k is optimal if $\lambda \geq a_k$.

[1]For those of you who are interested, these LPs were the basis of the paper by Trevisan, Sorkin, Sudan, and Williamson, "Gadgets, Approximation, and Linear Programming," *SIAM Journal on Computing* 29:2074-2097, 2000.