ORIE 6300 Mathematical Programming I	September 19, 2014

Problem Set 4

Due Date: September 26, 2014

1. In class we showed that given a polytope $Q = conv(v_1, \ldots, v_k)$, if 0 is in the interior of Q, then Q is a bounded polyhedron. Now suppose that we only know that there is some point v in the interior of Q. Show that Q is bounded polyhedron. (Hint: Think about $Q - v = \{w - v : w \in Q\}$).

2. Consider the set $P = \{x : Ax \geq 0\}$ and assume that we have $x \geq 0$ for all $x \in P$, i.e., that $x \geq 0$ is implied by $Ax \geq 0$.

   (a) A set K is a cone if $x, y \in K$ implies that $\lambda x + \mu y \in K$ for all $\mu, \lambda \geq 0$. Prove that P is a cone.

   (b) An *extreme ray* of a cone K is a nonzero vector $x \in K$ such that $x + y \in K$ and $x - y \in K$ implies that $y = \lambda x$ for some λ.

   Give another characterization of the extreme rays of the polyhedral cone P, using the rank of a submatrix of A. (Hint: think about the positive orthant as the canonical example of a cone, in order to get some intuition here.)

   (c) Two extreme rays x and y of a cone K are said to be the same if $x = \lambda y$ for some $\lambda > 0$. Prove that the number of different extreme rays of our polyhedral cone P is finite. Give a finite bound on the maximum number of extreme rays possible assuming that A is has m rows and n columns.

   (d) Let $r^1, \ldots, r^k$ denote the finite set of extreme rays of P. Let

   $$Q = cone(r^1, \ldots, r^k) = \{x = \sum_i \lambda_i r^i : \lambda_i \geq 0 \text{ for all } i\}.$$

   Prove that $P = Q$. (Hint: consider $P' = \{x \in P : \sum x_i = 1\}$.)

   It might help to visualize this as moving from the description of P by the faces of the cone that bound it ($Ax \geq 0$) to a description of P by the outside rays $(r^1, \ldots, r^k)$ that bound it.

3. Fun with polars.

   (a) Given a convex cone $K \subseteq \Re^n$, prove that the polar of K is the set $\{z \in \Re^n : x^T z \leq 0 \text{ for all } x \in K\}$.

   (b) Give the polar of the non-negative orthant $\{x \in \Re^n : x \geq 0\}$.

   (c) Show that if $A \subseteq B$, then $B^\circ \subseteq A^\circ$.