ORIE 6300 Mathematical Programming I	December 2, 2014

Lecture 27

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1 Conic Programming and Strong Duality

Recall from last time on primal/dual pair of conic programming

$$\begin{array}{llll} \inf & c^T x & \sup & b^T y \\ s.t. & Ax = b & s.t. & A^T y + s = c \\ & x \in K & & s \in K^* \equiv \{s \in \mathbb{R}^n, \quad s^T x \le 0, \quad \forall x \in K\} \end{array}$$

Various weirdness emerge even from "nice" cones (like SOC):

- Weak duality holds;
- Strong duality may not;
- Maybe no optimal solution(hence inf/sup)

Today we work out some conditions under which strong duality holds. Recall we showed the analog of Farkas' Lemma doesn't hold. Both $Ax = b, \quad x \in K$ and $-A^T y \in K^*, b^T y > 0$ can be infeasible. So let's see how we can modify the condition so we can get something.

Definition 1 *Ax=b, $x \in K$ is asymptotically feasible if $\forall \epsilon > 0$. $\exists \Delta b, \|\Delta b\| < \epsilon$ such that $Ax = b + \Delta b, x \in K$ is feasible.*

Theorem 1 *(Asymptotic Farkas' Lemma) Either $Ax = b, x \in K$ is asymptotically feasible, or $-A^T y \in K^x, b^T y > 0$ is feasible, but not both.*

Proof: Let $Q = \{\tilde{b} \in \mathbb{R}^m : \exists x \in K \text{ s.t. } Ax = \tilde{b}\}$, Then $b \in cl(Q)$ iff $Ax = b, x \in K$ asymptotically feasible (where $cl(Q)$ is the closure of Q). If $Ax = b, x \in K$ not asymptotically feasible, then $b \notin cl(Q)$. Since cl(Q) is closed, nonempty ($0 \in Q$), and convex, we can apply the separating hyperplane theorem. So $\exists y \in \mathbb{R}^n$, β such that $y^T b > \beta$ and $y^T \tilde{b} < \beta$ for all $\tilde{b} \in cl(Q)$. Since $0 \in Q$, $\beta > 0$. Thus

$$\begin{array}{lll} y^T b > 0 \quad \text{and} & & y^T (Ax) < \beta, \quad \forall x \in K \\ & \Leftrightarrow & y^T (A\lambda x) < \beta, \quad \forall x \in K, \quad \lambda > 0 \\ & \Leftrightarrow & y^T (Ax) < \beta/\lambda, \quad \forall x \in K, \quad \lambda > 0 \\ & \Leftrightarrow & y^T (Ax) \le 0, \quad \forall x \in K \\ & \Leftrightarrow & x^T (A^T y) \le 0, \quad \forall x \in K \\ & \Leftrightarrow & -A^T y \in K^*, \end{array}$$

by the definition of K^*. $\square$

Now we need to define the primal value under asymptotic feasibility.

Definition 2 *a-opt* $= \lim_{\varepsilon\to 0} \inf_{\|\Delta b\|} \inf\{c^T x : x \in K, Ax = b + \Delta b\}$ *(i.e. limiting value of asymptotically feasible solution)*

Theorem 2 *If primal is asymptotically feasible, then a-opt equals dual optimal.*

Proof: Consider the following system:

$$\begin{bmatrix} A & 0 \\ c^T & 1 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} b \\ \lambda \end{bmatrix}, \quad \begin{bmatrix} x \\ z \end{bmatrix} \in K \times \mathbb{R}^{\geq 0} \tag{$*$}$$

$$\begin{aligned} \Leftrightarrow Ax &= b \\ c^T x + z &= \lambda; \quad z \geq 0 \quad (i.e. \quad c^T x \leq \lambda) \end{aligned}$$

By asymptotic Farkas Lemma, either the system above is asymptotically feasible, and thus a-opt $\leq \lambda$ or the system

$$-\begin{bmatrix} A & 0 \\ c^T & 1 \end{bmatrix}^T \begin{bmatrix} y \\ \gamma \end{bmatrix} \in (K \times \mathbb{R}^{\geq 0})^* = K^* \times \mathbb{R}^{\geq 0}$$

$$\begin{bmatrix} b \\ \lambda \end{bmatrix}^T \begin{bmatrix} y \\ \gamma \end{bmatrix} > 0$$

is feasible which means that there exists y, γ such that

$$\begin{aligned} A^T y - \gamma c &\in K^* \\ -\gamma &\geq 0 \\ b^T y &> -\lambda\gamma \end{aligned} \tag{$**$}$$

First, suppose that there is λ such that a-opt $\leq \lambda <$ dual-optimal. Then there exists a dual feasible y such that $b^T y > \lambda$, $-A^T y + c \in K^*$. But then $\begin{bmatrix} y \\ -1 \end{bmatrix}$ is feasible for (**), a condradiction to a-opt $\leq \lambda$. Thus a-opt $\geq$ dual-optimal.

Second, suppose that there is a λ such that dual-opt $< \lambda <$a-opt. Then (**) is feasible for same $\begin{bmatrix} y \\ \gamma \end{bmatrix}$. If $\gamma = 0$ then $-A^T y \in K^*$, $b^T y > 0$ is feasible. By the asymptotic Farkas' Lemma, this implies that $Ax = b,\ x \in K$ is not asymptotically feasible, which is a contradiction.

Thus we can assume $\gamma < 0$. Then consider $\tilde{y} = -\frac{1}{\gamma} y$. By the feasibility of (y, γ), we have that

$$\begin{aligned} -A^T \tilde{y} + c &\in K^* \\ b^T \tilde{y} &> \lambda. \end{aligned}$$

This contradicts our hypothesis that dual-optimal $< \lambda$.

We have shown that a-opt can be neither less than the dual optimal, nor greater than the dual optimal, and so it must be equal to the dual optimal. $\square$

We can similarly define the asymptotic optimal of dual.

Definition 3 *a-dual-opt* $= \lim_{\epsilon \to 0} \sup_{\|\Delta c\| < \epsilon} (\sup b^T y : c + \Delta c - A^T y \in K^*)$.

By similar reasoning, we can prove the following theorem.

Theorem 3 *If dual is asymptotic feasible, then a-dual-opt = primal opt.*

We now want to state conditions under which strong duality holds. We now know that a-opt = dual optimal $\leq$ a-dual-opt = primal optimal. When is the inequality an equality? We first need another definition.

Definition 4 *The primal is strongly feasible if* $\exists \epsilon > 0$ *such that* $\forall \Delta b$ *with* $\|\Delta b\| < \epsilon$*, then* $Ax = b + \Delta b, x \in K$ *is feasible.*

The dual is strongly feasible if $\exists \epsilon > 0$ *such that* $\forall \Delta c$ *with* $\|\Delta c\| < \epsilon$ *, then* $A^T + S = c + \Delta c, s \in K^*$ *is feasible.*

Observation 1 *If* $\exists x$ *such that* $Ax = b, x \in$ *int* K*, then the primal is strongly feasible.*

Theorem 4 *If either primal or dual is strongly feasible, then primal opt = dual opt. (i.e. strong duality holds).*

Corollary 5 *Strong duality holds if there exists a feasible primal solution* $x \in$ *int* K*, or if there exists a feasible dual solution with* $s \in$ *int* K^**.*

Proof: Assume primal, dual are both asymptotic feasible, and the dual is strongly feasible. (Note that we are skipping a case in which the primal is infeasible, the dual unbounded). Then,

$$\text{a-opt} = \text{dual} \leq \text{a-dual-opt} = \text{primal}.$$

Suppose that a-opt < primal. Then, there exists a sequence $\{x_i\} \in K$ and $\{\Delta b_i\}$ such that

$$Ax_i = b + \Delta b_i, \quad \Delta b_i \to 0, \quad c^T x_i \to \text{a-opt}.$$

We claim that $\|x_i\| \to \infty$, since otherwise $\{x_i\}$ has convergent subsequence, with limit point x having $x \in K$, $Ax = b$, and $c^T x =$a-opt. Such an x implies that a-opt = primal.

Let Δc be a limit point of $\{-\frac{x_i}{\|x_i\|}\}$ so that $\|\Delta c\| = 1$. For given $\epsilon > 0$ consider

$$\begin{aligned} \min \quad & (c + \epsilon \Delta c)^T x \\ s.t. \quad & Ax = b \\ & x \in K. \end{aligned}$$

The asymptotic optimal of this instance is at most

$$\begin{aligned}
\liminf_i (c + \epsilon\Delta c)^T x_i &= \liminf_i c^T x_i + \epsilon \liminf_i \Delta c^T x_i \\
&= \text{a-opt} + \epsilon \liminf_i \Delta c^T x_i \\
&= \text{a-opt} - \epsilon \lim_i \|x_i\| \\
&= -\infty.
\end{aligned}$$

Since the asymptotic optimal is unbounded, dual must be infeasible; i.e. the following is infeasible:

$$\begin{aligned}
&\sup \quad b^T y \\
&A^T y + s = c + \epsilon\Delta c \\
&s \in K^*
\end{aligned}$$

Since ϵ can be arbitrarily small, this implies that the original dual is not strongly feasible, a contradiction. □

Corollary 6 *If primal is feasible and dual is strongly feasible, then the primal has an optimal solution.*

Proof: As above. If the dual is strongly feasible and the primal feasible, then strong duality holds and there exists a feasible sequence $\{x_i\} \subset K, \quad \Delta b_i \to 0, \quad c^T x_i \to$ a-opt. If$\{x_i\}$ does not have a convergent subsequence, then $\|x_i\| \to \infty$ implies that the dual is not strongly feasible. So there is a convergent subsequence, and the limit point x is feasible, with $c^T x = opt$. □