ORIE 634 Approximation Algorithms October 17, 2006

Problem Set 4

Due Date: November 2, 2006

1. Show that the performance guarantee of the primal-dual algorithm for the uncapacitated facility location algorithm given in class can be strengthened in the following way. Suppose that the algorithm opens the set T' of facilities and constructs the dual solution (v, w). Show that

$$\sum_{j \in D} \min_{i \in T'} c_{ij} + 3 \sum_{i \in T'} f_i \leq 3 \sum_{j \in D} v_j.$$

2. Recall that when we discussed the prize-collecting Steiner tree problem, we used the following linear programming relaxation:

$$\begin{array}{llll} \text{Min } \displaystyle\sum_{e \in E} c_e x_e + \sum_{i \in V} \pi_i (1 - y_i) & & & \\ \displaystyle\sum_{e \in \delta(S)} x_e & \geq & y_i & \forall i \in S, \forall S \subseteq V - r, S \neq \emptyset \\ y_r & = & 1 & \\ y_i & \geq & 0 & \forall i \in V \\ x_e & \geq & 0 & \forall e \in E. \end{array}$$

   Let (x, y) be any feasible solution to this linear program. We claimed that given a fixed value of $\alpha > 0$, we could construct a tree T spanning at least those vertices with $y_i > \alpha$ such that

$$\sum_{e \in T} c_e \leq \frac{2}{\alpha} \sum_{e \in E} c_e x_e.$$

   Show that this can be done using the primal-dual method. (Hint: this can be done without devising a new algorithm).

3. Consider the minimum multicut problem in trees. In this problem, we are given a tree $T = (V, E)$, k pairs of vertices $s_i - t_i$, and costs $c_e \geq 0$ for each edge $e \in E$. The goal is to find a minimum-cost set of edges F such that for all i, s_i and t_i are in different connected components of $G' = (V, E - F)$.

Let P_i be the set of edges in the unique path in T between s_i and t_i. Then we can formulate the problem as the following integer program:

$$\begin{aligned} \text{Min} \sum_{e \in E} c_e x_e & & & \\ \sum_{e \in P_i} x_e & \geq 1 & & 1 \leq i \leq k \\ x_e & \in \{0,1\} & & e \in E. \end{aligned}$$

Suppose we root the tree at an arbitrary vertex r. Let $depth(v)$ be the number of edges on the path from v to r. Let $lca(s_i, t_i)$ be the vertex v on the path from s_i to t_i whose depth is minimum. Suppose we use the primal-dual method to solve this problem, where the violated constraint we choose in each iteration is the one that maximizes $depth(lca(s_i, t_i))$.

Prove that this gives a 2-approximation algorithm for the minimum multicut problem in trees.

4. In the k-path partition problem, we are given a complete, undirected graph $G = (V, E)$ with edge costs $c_e \geq 0$ that obey the triangle inequality (that is, $c_{(u,v)} \leq c_{(u,w)} + c_{(w,v)}$ for all $u, v, w \in V$), and a parameter k such that $|V|$ is a multiple of k. The goal is to find a minimum-cost collection of paths of k edges each such that each vertex is on exactly one path.

   A related problem is that of partitioning a graph into $0 (\bmod k)$-trees. The input to this problem is the same as that above, except that the graph is not necessarily complete and edge-costs do not necessarily obey the triangle inequality. The goal is to find a minimum-cost collection of trees such that each tree has $0 (\bmod k)$ vertices, and each vertex is in exactly one tree.

   (a) Given an α-approximation algorithm for the second problem, produce a $2\alpha\left(1 - \frac{1}{k}\right)$-approximation algorithm for the first.

   (b) Use the primal-dual method to give a 2-approximation algorithm for the second problem.

   (c) Give a $4\left(1 - \frac{1}{k}\right)$-approximation algorithm for the problem of partitioning a graph into *cycles* of length exactly k.

5. In class we gave a primal-dual $O(\log n)$-approximation algorithm for the feedback vertex set problem in undirected graphs. This problem asks you to give a primal-dual 2-approximation algorithm for the problem by using a different integer programming formulation of the problem.

(a) Show that for any feedback vertex set of the graph $G = (V, E)$,

$$\sum_{v \in F} (d(v) - 1) \geq |E| - |V| + 1,$$

where $d(v)$ is the degree of vertex v in the graph.

(b) Let $G[S] = (S, E[S])$ be the graph induced by considering the subset $S \subseteq V$ of vertices, where $E[S]$ is the subset of edges that have both endpoints in S. Let $d_S(v)$ be the degree of v in $G[S]$. Argue that the following integer program models the feedback vertex set problem in undirected graphs.

$$\begin{aligned} \text{Min } \sum_{i \in V} w_i x_i & & & \\ \sum_{i \in S} (d_S(i) - 1) x_i & \geq & |E[S]| - |S| + 1 & \qquad \forall S \subseteq V, S \neq \emptyset \\ x_i & \in & \{0, 1\}. & \end{aligned}$$

(c) (Bonus problem) Use the formulation above to give a primal-dual 2-approximation algorithm for the feedback vertex set problem in undirected graphs.