ORIE 634 Approximation Algorithms September 14, 2006

Problem Set 2

Due Date: September 28, 2006

1. The *k-suppliers* problem is similar to the k-center problem. The input to the problem is a positive integer k, and a set of vertices V in a metric space, along with distances d_{ij} between any two vertices i, j that obey the same properties as in the k-center problem. However, now the vertices are partitioned into *suppliers* $F \subseteq V$ and *customers* $D = V - F$. The goal is to find k suppliers such that the maximum distance from a supplier to a customer is minimized. In other words, we wish to find $S \subseteq F$, $|S| \leq k$, that minimizes $\max_{j \in C} d(j, S)$. Give a 3-approximation algorithm for this problem.

2. Let E be a set of items, and for $S \subseteq E$, let $f(S)$ give the value of the subset S. Suppose we wish to find a maximum value subset of E of at most k items. Furthermore, suppose that f is *monotone* and *submodular*. We say that f is monotone if for any S and T with $S \subseteq T \subseteq E$, then $f(S) \leq f(T)$. We say that f is *submodular* if for any $S, T \subseteq E$, then

$$f(S) + f(T) \geq f(S \cup T) + f(S \cap T).$$

   (a) Show that the greedy $(1 - \frac{1}{e})$-approximation algorithm we gave for float maximization extends to this problem.

   (b) (Bonus problem) Suppose that associated with each $e \in E$, there is a weight $w_e \geq 0$, and that $f'(S) = f(S) - \sum_{e \in E} w_e$. Give a $(1 - \frac{1}{e})$-approximation algorithm for finding a subset $S \subseteq E$, $|S| \leq k$, that maximizes the value $f'(S)$.

3. In the Karmarkar-Karp algorithm for the bin packing problem, we used *geometric grouping* in each iteration of the algorithm to create an instance I' from I. Consider the following grouping scheme: for $i = 0, \ldots, \lceil \log_2 SIZE(I) \rceil$, create a group G_i such that all pieces from I of size $(2^{-(i+1)}, 2^{-i}]$ are placed in group G_i. In each group G_i, create subgroups $G_{i,1}, G_{i,2}, \ldots, G_{i,k_i}$ by arranging the pieces in G_i from largest to smallest and putting the $4 \cdot 2^i$ largest in the first subgroup, the next $4 \cdot 2^i$ in the next subgroup, and so on. Now create instance I' from I in the following manner: for each $i = 0, \ldots, \lceil \log_2 SIZE(I) \rceil$, discard subgroups $G_{i,1}$ and G_{i,k_i} (i.e., the first and last subgroups of G_i) and for $j = 2, \ldots, k_i - 1$ round each piece of subgroup $G_{i,j}$ to the size of the largest piece in $G_{i,j}$.

   Prove that by using this grouping scheme within the Karmarkar-Karp algorithm, we obtain an algorithm KK' such that $KK'(I) \leq OPT(I) + O(\log^2(OPT(I)))$ for all instances I.

4. (Bonus problem) In class, we stated that it is possible to solve the bin packing linear program to within additive error of 1 in time polynomial in n (the number of items) and $\log(m/s_m)$ (where m is the number of item sizes and s_m is the size of the smallest item). Prove this theorem.

5. Consider the linear programming relaxation for the MAX SAT problem that we saw in class:

$$
\begin{array}{lll}
\text{Max} & \displaystyle\sum_{j=1}^{m} w_j z_j & \\
& \displaystyle\sum_{i \in P_j} y_i + \sum_{i \in N_j} (1 - y_i) \geq z_j & \displaystyle\forall C_j = \bigvee_{i \in P_j} x_i \bigvee_{i \in N_j} \bar{x}_i \\
& 0 \leq y_i \leq 1 & 1 \leq i \leq n \\
& 0 \leq z_j \leq 1 & 1 \leq j \leq m.
\end{array}
$$

Prove that using randomized rounding with the linear function $f(y_i) = \frac{1}{2}y_i + \frac{1}{4}$ also gives a $\frac{3}{4}$-approximation algorithm for MAX SAT.