ORIE 634 Approximation Algorithms August 31, 2006

Problem Set 1

Due Date: September 14, 2006

1. Prove that it is NP-hard to approximate the k-center problem to within a factor of $2 - \epsilon$ for any $\epsilon > 0$. It might be helpful to use the fact that the dominating set problem is NP-complete. In the dominating set problem, we are given an undirected graph $G = (V, E)$, and a bound B, and must decide whether there exists a dominating set $S \subseteq V$ of size at most B (that is, $|S| \leq B$). A set S of vertices is said to be a dominating set if for all $i \in V$, either $i \in S$ or there exists an edge (i, j) and $j \in S$.

2. Consider the weighted vertex cover problem.

   (a) Prove that any extreme point of the linear program

$$\text{Min} \quad \sum_{i \in V} w_i x_i$$

subject to:

$$\begin{aligned} x_i + x_j &\geq 1 & \quad \forall (i,j) \in E \\ x_i &\geq 0 & \quad i \in V \end{aligned}$$

   has the property that $x_i \in \{0, \frac{1}{2}, 1\}$ for all i (Recall that an extreme point x is a feasible solution that cannot be expressed as $\lambda x^1 + (1-\lambda)x^2$ for $0 < \lambda < 1$ and feasible solutions x^1 and x^2 distinct from x.)

   (b) Give a $\frac{3}{2}$-approximation algorithm for the weighted vertex cover problem when the input graph is planar. You may use the fact that polynomial-time LP solvers return extreme points, and that there is a polynomial-time algorithm to 4-color any planar graph (i.e. the algorithm assigns each vertex one of four colors such that for any edge $(i, j) \in E$, vertices i and j have been assigned different colors).

3. In the weighted Set Cover problem, one finds a collection I of sets that minimizes $\sum_{j \in I} w_j$ such that

$$|\bigcup_{j \in I} S_j| = |T|.$$

   Consider the partial cover problem, in which one finds a collection I of sets that minimizes $\sum_{j \in I} w_j$ such that

$$|\bigcup_{j \in I} S_j| \geq p|T|,$$

   where $0 < p < 1$ is some constant.

(a) Give a polynomial-time algorithm to find a solution to the partial cover problem in which the value is no more than $c(p) \cdot \mathrm{OPT}$, where $c(p)$ is a constant that depends on p, and OPT is the value of the optimal solution to the set cover problem.

(b) (Bonus problem) Give an $f(p)$-approximation algorithm for the partial cover problem, such that f is nondecreasing in p and $f(1) \leq H_{|T|}$.

4. Consider the following problem:

Directed Steiner Tree

- **Input:**
  - A directed graph $G = (V, E)$
  - Costs $c_{ij} \geq 0$ for arcs $(i, j) \in E$
  - A root vertex $r \in V$
  - A set of terminal $T \subseteq V$
- **Goal:** Find a minimum-cost tree such that for each $i \in T$ there exsts a directed path from r to i.

Prove that for some constant c there can be no $c \log |T|$-approximation algorithm for the Directed Steiner Tree problem unless $P = NP$.

(Research problem: Give a $c \log^k |T|$-approximation algorithm for the Directed Steiner Tree problem for some constants c and k).

5. In the *asymmetric traveling salesman problem with triangle inequality*, we are given as input a complete directed graph $G = (V, A)$ with costs $c_{ij} \geq 0$ for all arcs $(i, j) \in A$, such that the arc costs obey the *triangle inequality*: for all $i, j, k \in V$, we have that $c_{ij} + c_{jk} \geq c_{ik}$. The goal is to find a *tour* of minimum cost; that is, a directed cycle that contains each vertex exactly once, such that the sum of the cost of the arcs in the cycle is minimized.

One approach to finding an approximation algorithm for this problem is to first find a minimum-cost strongly connected *Eulerian* subgraph of the input graph. A directed graph is strongly connected if for any pair of vertices $i, j \in V$ there is a path from i to j and a path from j to i. A directed graph is Eulerian if the indegree of each vertex equals its outdegree. Given strongly connected Eulerian subgraph of the input to the problem, it is possible to use a technique called "shortcutting" to turn this into a tour of no greater cost by using the triangle inequality.

One way to find a strongly connected Eulerian subgraph is as follows: we first find a *minimum mean-cost cycle* in the graph. A minimum-mean cost cycle is a directed cycle that minimizes the ratio of the cost of the arcs in the cycle to the number of arcs in the cycle. Such a cycle can be found in polynomial time.

We then choose one vertex of the cycle arbitrarily, remove all other vertices of the cycle from the graph, and repeat. We do this until only one vertex of the graph is left.

(a) Prove that this algorithm finds a strongly connected Eulerian directed subgraph of the input graph.

(b) Prove that the cost of this subgraph is at most $2H_n \cdot \mathrm{OPT}$, where $n = |V|$ and OPT is the cost of the optimal tour. Conclude that this algorithm is a $2H_n$-approximation algorithm for the asymmetric traveling salesman problem with triangle inequality.

6. (Bonus problem) In the *node-weighted Steiner tree* problem, we are given as input an undirected graph $G = (V, E)$, node weights $w_i \geq 0$ for all $i \in V$, edge costs $c_e \geq 0$ for all $e \in E$, and a set of terminals $T \subseteq V$. The cost of a tree is the sum of the weights of the nodes plus the sum of the costs of the edges in the tree. The goal of the problem is to find a minimum-weight tree that spans all the terminals in T.

   Give a greedy $O(\ln n)$-approximation algorithm for the node-weighted Steiner tree problem.