CONDITION NUMBERS AND GENERALIZED EQUATIONS

Adrian Lewis

Cornell University

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orie.cornell.edu/∼aslewis

Technion: Lecture 3

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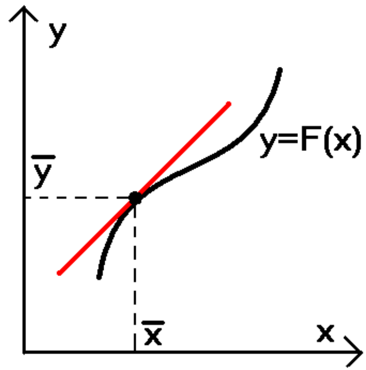

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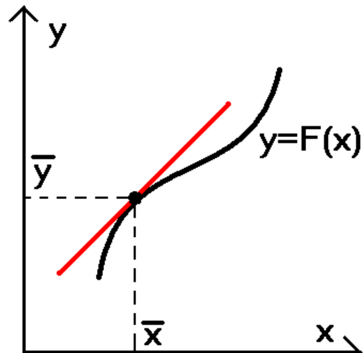

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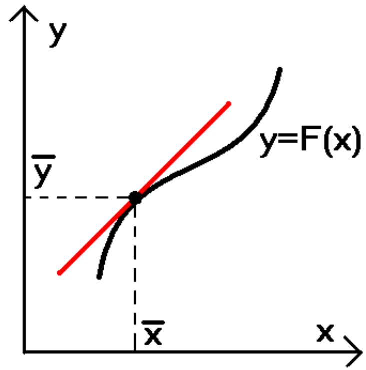

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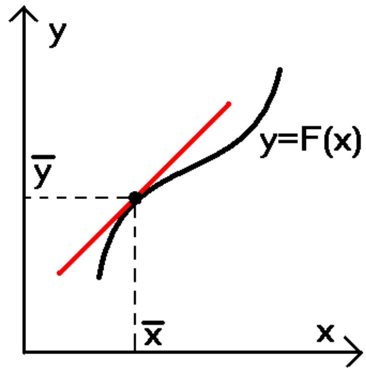

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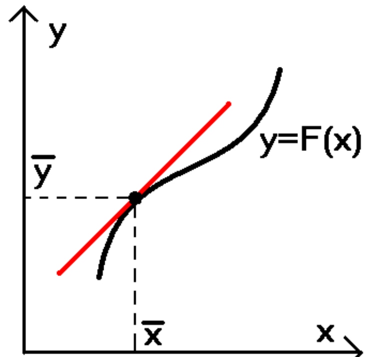

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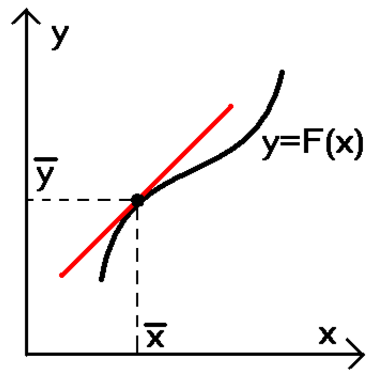

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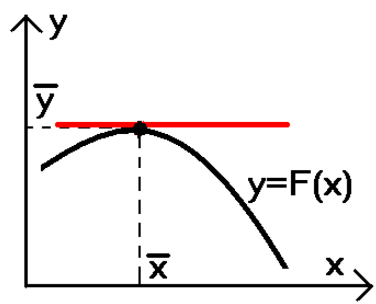

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Furthermore, $\operatorname{reg} F(\overline{x})$ governs the **speed** of natural iterative schemes (like conjugate gradients) for solving $F(x) = \overline{y}$.

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(and the minimum is attained by rank-one G).

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To summarize, for smooth $F : \mathbf{R}^n \to \mathbf{R}^m$, the regularity modulus and radius at $\overline{x}$ are **reciprocals**:

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- these quantities influence the speed of solution algorithms.

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The previous ideas apply: **set-valued** $F : \mathbf{R}^n \rightrightarrows \mathbf{R}^m$ is **regular** at $(\bar{x}, \bar{y})$ (with $\bar{y} \in F(\bar{x})$) if a linear error bound holds:

$$d(x, F^{-1}(y)) \leq k d(y, F(x)) \;\text{ for all } (x, y) \text{ near } (\bar{x}, \bar{y}).$$

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We can study systems of **inequalities** like

$$Ax \le y, \quad x \ge 0$$

via the **generalized equation** $y \in F(x)$, where

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$$\operatorname{rad} F(\overline{x}|\overline{y}) \;=\; \inf_{\text{linear } G} \{\|G\| : F + G \text{ irregular at } (\overline{x}, \overline{y} + G\overline{x})\}.$$

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Theorem Consider set-valued $F : X \rightrightarrows Y$ between Banach spaces, with closed convex **graph**

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So, we need to understand surjectivity of sublinear mappings.

10. RENEGAR'S THEORY

Recall the Eckart-Young formula for linear maps

$$\min_T \{\|T\| : (A+T)\mathbf{R}^n \neq \mathbf{R}^m\}$$

$$= \min_{\|v\|=1} \sup_x \left\{ \frac{1}{\|x\|} : v = Ax \right\}.$$

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Renegar (1995) extended this:

$$\min_T \{\|T\| : (A+T)K + L \neq \mathbf{R}^m\}$$
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Renegar also showed how the radius governs the speed of natural interior point algorithms for conic convex programs.

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Interpretation: Small perturbations can render F nonsurjective exactly when some unit vector v forces all solutions of $v \in F(x)$ to be large.

12. THE RECIPROCAL FORMULA IN GENERAL

Theorem (Dontchev-Lewis-Rockafellar 2003) For **any** closed set-valued mapping $F : \mathbf{R}^n \to \mathbf{R}^m$ with $\bar{y} \in F(\bar{x})$,

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- The domain can be Riemannian (Dontchev-Lewis 2005).

14. STRUCTURED SINGULAR VALUES

For a square matrix A, we've seen

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Can we characterize μ?

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